Symmetry reductions of the Skyrme-Faddeev model

L. Martina

Dip. Matematica e Fisica "E. De Giorgi" - Univ. Salento, Sezione INFN, Lecce, Italy

Geometry of PDEs and Integrability
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References

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The Skyrme Faddeev- Model

$$E\left[\boldsymbol{n}\right] = \int_{\mathbb{R}^3} \left\{ (\partial_a \boldsymbol{n})^2 + \frac{\lambda}{2} \left((\boldsymbol{n} \cdot \partial_a \boldsymbol{n} \times \partial_b \boldsymbol{n}) \right)^2 \right\} \mathrm{d}^3 x, \quad \boldsymbol{n} \in \mathbb{S}^2$$

$$\begin{array}{lcl} E_1 = \int_{\mathbb{R}^3} (\partial_a \boldsymbol{n})^2 \, \mathrm{d}^3 x & & \Lambda \, E_1 \\ & \boldsymbol{x} \to \frac{\boldsymbol{x}}{\Lambda} \Rightarrow & \\ E_2 = \int_{\mathbb{R}^3} \left(\frac{\lambda}{2} \boldsymbol{n} \cdot \partial_a \boldsymbol{n} \times \partial_b \boldsymbol{n} \right)^2 \mathrm{d}^3 x & & \Lambda^{-1} \, E_2 \end{array}$$

Derrick' Th. Stable Static Solutions $\Rightarrow \; \Lambda = \sqrt{\frac{E_2}{E_1}} \approx \sqrt{\lambda}$

The Hopf Charge and Hopfions

$$\lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{n}(\boldsymbol{x}) = \boldsymbol{n}_\infty = \pm \boldsymbol{z} \quad \Rightarrow \quad \boldsymbol{n} : \mathbb{S}^3 \to \mathbb{S}^2$$

$$\pi_3\left(\mathbb{S}^2\right) = \mathbb{Z}$$

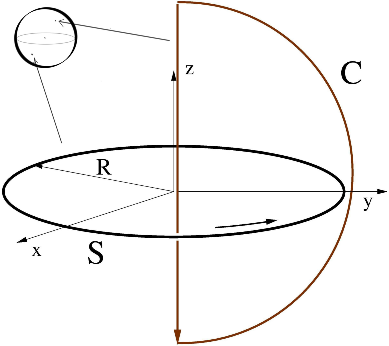
$$\mathcal{H} = \frac{1}{2}\left(\boldsymbol{n}\cdot\partial_a \boldsymbol{n} \times \partial_b \boldsymbol{n}\right) \mathrm{d}x_a \wedge \mathrm{d}x_b : \quad \mathrm{d}\,\mathcal{H} = 0$$

$$H^2\left(\mathbb{S}^3\right) = \{0\} \Rightarrow \mathcal{A} = \mathcal{A}_a\,\mathrm{d}x_a \ : \ \mathcal{H} = \mathrm{d}\mathcal{A}$$

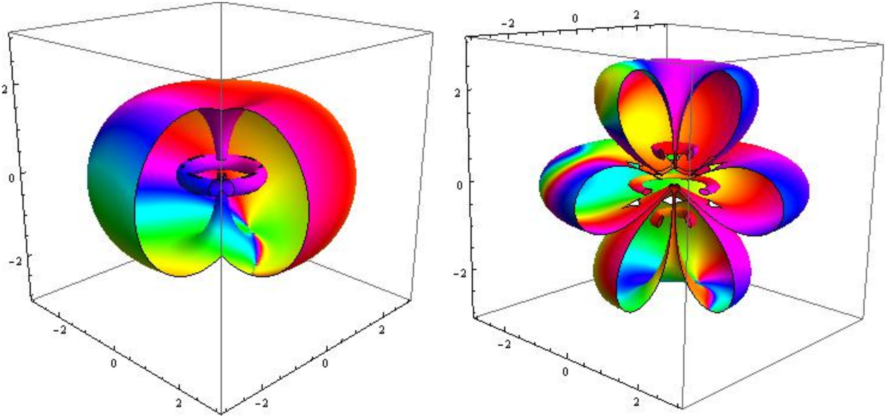
The Hopf Invariant

$$N[\boldsymbol{n}] = \frac{1}{4\pi^2}\int_{\mathbb{S}^3} \mathcal{H}\wedge\mathcal{A}$$

$n_1 + \mathrm{i} n_2 =$
$\mathrm{e}^{\mathrm{i}(m\phi(r,z) - n\psi(r,z))} \sin\Theta\,(r,z)$
$n_3 = \cos\Theta\,(r,z)$
$C : \boldsymbol{n} = \boldsymbol{n}_\infty \;\; S : \boldsymbol{n} = -\boldsymbol{n}_\infty$



$$
\begin{aligned}
\mathcal{A} &= n\cos^2\frac{\Theta}{2}\,\mathrm{d}\psi + m\sin^2\frac{\Theta}{2}\,\mathrm{d}\phi \\
\mathcal{H}\wedge\mathcal{A} &= nm\cos^2\frac{\Theta}{2}\sin\Theta\,\mathrm{d}\psi\wedge\mathrm{d}\Theta\wedge\mathrm{d}\phi \\
N[\boldsymbol{n}] &= m\,n
\end{aligned}
$$



Level sets $n_3 = 0.9$ and $n_3 = -0.9$ for maps with $N[\boldsymbol{n}] = 1$ (left) and $N[\boldsymbol{n}] = 3$ (right)

The energy bound

$$E\left[\boldsymbol{n}\right] \geq c\,\sqrt{\frac{\lambda}{2}}\,\left|N\left[\boldsymbol{n}\right]\right|^{3/4}, \quad c \approx (3/16)^{3/8}$$

A. F. Vakulenko, L. V. Kapitansky, *Sov. Phys. Dokl.* **24** (1979); 433 A. Kundu, Y. P. Rybakov, *J. Phys. A* **15** (1982), 269; J. Gladikowski, M. Hellmund, Phys. Rev. D 56, 5194 (1997)

R. S. Ward, *Nonlinearity* **12** (1999), 241 $c = 1$ (*conjecture*)

P. Sutcliffe, *Proc. R. Soc. A* **463** (2007), 3001 *Links and Knots of higher topological charge*

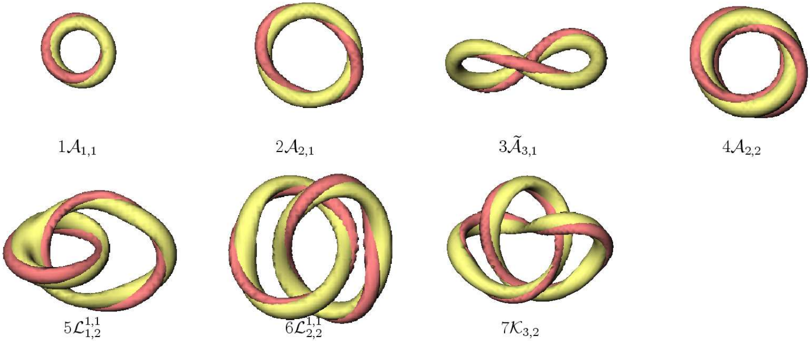
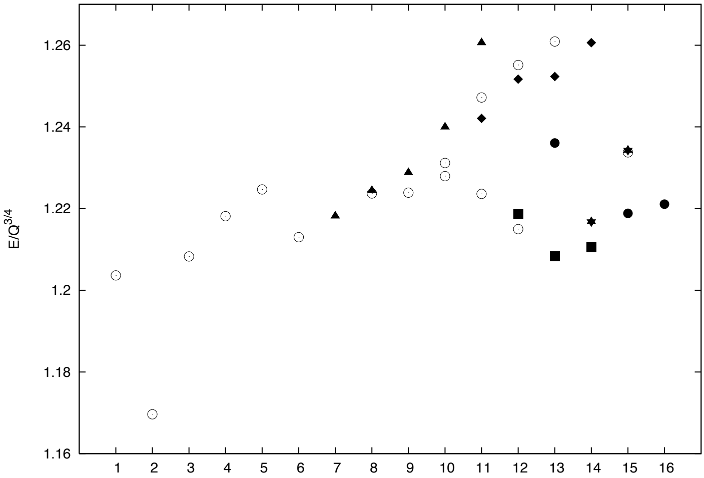
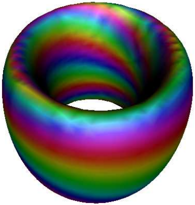


Figure 1: The position (light tube) and linking (dark tube) curves for the known lowest energy solitons with Hopf charges $1 \leq Q \leq 7$.

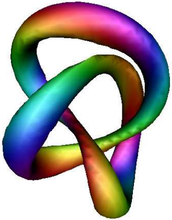
(*Sutcliffe '07*)



Ratio $E\,[\boldsymbol{n}]\,/\,(N\,[\boldsymbol{n}])^{\frac{3}{4}}$ as function of N: unknots and links (circles), knots (other signs) (*Sutcliffe '07*)



(a) Initial state of the knot Hopfion



(b) Final state of the knot Hopfion

Relaxation of a N=7 distribution (*Jäykkä '09*)
L. D. Faddeev e A. J. Niemi, *Nature* **387** (1997), 58

In Matter Physics

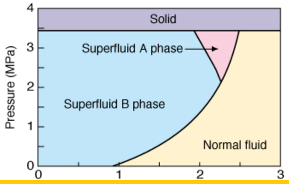
1. $^3He - A$ superfluid $(M_L = 1, M_S = 0)$
2. 2-band superconductor (Nb-doped $SrTiO_3$, MgB_2)
3. charged condensates of tightly bounded fermion pairs
4. Dzyaloshinskii Moriya interaction Magnets $S_1 \cdot S_2 \times S_3$

$$\eta_{\mu i} = \Delta^0 \hat{d}_\mu \left(\hat{m}_i^1 + \imath \hat{m}_i^2 \right), \qquad \hat{\mathbf{m}}^i \cdot \hat{\mathbf{m}}^j = \delta^{ij}, |\hat{\mathbf{d}}| = 1$$

$$G = U(1) \times SO(3)_L \times SO(3)_S \to H = U(1)_{L_z} \times U(1)_{S_z}$$

$$\text{superfluid velocity} \qquad \mathbf{v}_s = \frac{\hbar}{2m} \hat{m}_i^1 \nabla \hat{m}_i^2 \qquad \text{no vorticity quantization}$$

$$\text{Mermin - Ho vorticity} \qquad \nabla \times \mathbf{v}_s = \frac{\hbar}{4m} \epsilon_{ijk} \hat{l}_i \nabla \hat{l}_j \times \nabla \hat{l}_k \quad \hat{\mathbf{l}} = \hat{\mathbf{m}}^1 \times \hat{\mathbf{m}}^2$$



The 2comp - Ginzburg - Landau Model

$$\mathcal{E} = \int d^3x \left[\sum_\alpha \left(\frac{\hbar^2}{2m_\alpha} \left| \left(\partial_k + iqA_k \right) \Psi_\alpha \right|^2 + b_\alpha |\Psi_\alpha|^2 + \frac{c_\alpha}{2} |\Psi_\alpha|^4 \right) + \frac{\left(\nabla \times \mathbf{A} \right)^2}{8\pi} \right]$$

$$\Psi_\alpha = \sqrt{2m}\, \rho\, \chi_\alpha, \;\; \rho = \sum_\alpha \frac{|\Psi_\alpha|^2}{2m_\alpha} \;\; \chi_\alpha = |\chi_\alpha| e^{i\varphi_\alpha}, \quad |\chi_1|^2 + |\chi_2|^2 = 1$$

$$\chi = (\chi_1, \chi_2) \rightarrow \qquad \mathbf{n} = \chi^\dagger \boldsymbol{\sigma} \chi \qquad (\mathbf{n} \cdot \mathbf{n} = 1)$$

$$\begin{array}{ll} \mathbf{c} = 2\rho^2 \left(\mathbf{j} - 4\mathbf{A} \right) & \left\{ \begin{array}{ll} \mathbf{j} = i \sum_i \; \{ \chi_i \nabla \chi_i^* - c.c. \} & \text{paramagn. curr.} \\ -4\mathbf{A} & \text{diamagn. curr.} \end{array} \right. \\ \text{gauge invariant current} & \end{array}$$

$$\mathcal{E} = \int d^3x \left[\frac{1}{4} \rho^2 \left(\partial_k \mathbf{n} \right)^2 + \left(\partial_k \rho \right)^2 + \frac{1}{16} \rho^2 \mathbf{c}^2 + \left(F_{ik} - H_{ik} \right)^2 + V(\rho) \right]$$

$$F_{ik} = \partial_i c_k - \partial_k c_i$$

$$H_{ik} = \mathbf{n} \cdot [\partial_i \mathbf{n} \times \partial_k \mathbf{n}] := \partial_i a_k - \partial_k a_i \quad \text{Mermin} - \text{HoVorticity}$$

The Equations of Motion (static limit)

$$\mathbf{n} \times \bigtriangledown^2 \mathbf{n} \quad + \quad \frac{2}{\rho}\,\partial_k \rho\, \mathbf{n} \times \partial_k \mathbf{n} + \frac{8}{\rho^2}\,\partial_i H_{ik}\,\partial_k \mathbf{n} = \frac{2}{\rho^2}\,F_{ik}\,\partial_i \mathbf{n} \times \partial_k \mathbf{n} - \frac{4}{\rho^2}\partial_i\left(F_{ik}\mathbf{n} \times \partial_k \mathbf{n}\right) \quad ,$$

$$\nabla^2 \rho - \frac{1}{4}\left(\left(\partial_k \mathbf{n}\right)^2 + \frac{1}{2}\mathbf{c}^2\right)\rho = \frac{1}{2}V'(\rho) \quad ,$$

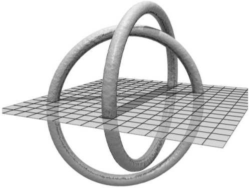
$$\partial_k F_{ki} - \partial_k H_{ki} = \frac{\rho^2}{32} c_i .$$

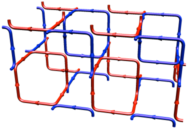
$O(3) -$ nonlinear σ model + new terms

Phases

- Skyrme-Faddeev model $\quad \mathbf{n} \neq const \quad \mathbf{c} = 0, \rho = const$
- 1c-GL model in e.m. $\quad \mathbf{n} = const, \quad \mathbf{c} \neq 0, \rho \neq const$
- Inhomogeneous Supercond. $\quad \mathbf{n} \neq const, \mathbf{c} \neq 0, \rho = const$
- quasi 1-D distributions (stripes) $\quad \mathbf{n} \neq const, \mathbf{c} = 0, \rho \approx \rho(f(\mathbf{r}))$

- Stability of the order parameter configurations
- Knotted and/or linked quasi-1-dimensional configurations
- Coexistence/Competition of short/long (UV/IR) wave modes
- Properties of knots and tangles
- Topological ordering in disordered background





L. Martina , A. Protogenov, V. Verbus,Theor. Math. Phys. **160**, n. (2009), 1058 - 1065; Theor. Math. Phys. 167(3) (2011), 843-855

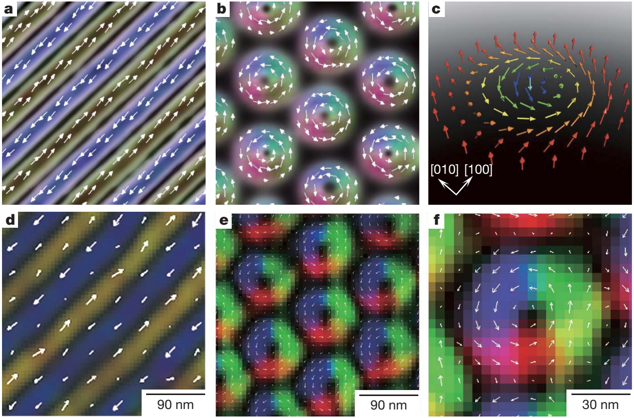


Figure 1 | Topological spin textures in the helical magnet $Fe_{0.5}Co_{0.5}Si$. **a**, **b**, Helical (**a**) and skyrmion (**b**) structures predicted by Monte Carlo simulation. **c**, Schematic of the spin configuration in a skyrmion. **d**–**f**, The experimentally observed real-space images of the spin texture, represented by the lateral magnetization distribution as obtained by TIE analysis of the Lorentz TEM data: helical structure at zero magnetic field (**d**), the crystal (SkX) structure for a weak magnetic field (50 mT) applied the thin plate (**e**) and a magnified view of **e** (**f**). The colour map arrows represent the magnetization direction at each point.

X. Z. Yu, Y. Onose, N. Nagaosa *et al* *Nature* **465** |17 (2010) 901

N. Romming *et al.*, *Science* , **341** n. 6146 (2013) 636

The Yang-Mills theory

- *mass gap* $\Leftrightarrow$ *short range interactions*
- *quark confinement* $\Leftrightarrow$ *adrons as color singlets bound states*

$SU(2)$ - Yang-Mills - No Matter

$$S = -\int \mathrm{tr}\, F \wedge \star F,$$

$$A = -\mathrm{i}\, T^a A^a_\mu(x)\, \mathrm{d}x^\mu, \quad T^a \in su(2), \qquad F = \mathrm{d}A + A \wedge A$$

$$\text{loc. g. inv.} \qquad A \to V^{-1} A\, V + V^{-1} \mathrm{d} V, \quad V \in SU(2)$$

$$\text{Field eq.s} \quad \mathrm{d} \star F + A \wedge \star F - \star F \wedge A = 0, \quad \mathrm{d}F + A \wedge F - F \wedge A = 0.$$

Faddeev-Niemi effective Lagrangian

L. D. Faddeev e A. J. Niemi, *Nucl. Phys. B* **776** (2007), 38

InfraRed limit + Quantum fluctuations : Spin-Charge Variables

$$
\begin{aligned}
\mathcal{L}^{eff}_{\mathrm{YM}}|_{\xi=1} &= \frac{1}{4}\mathcal{F}^2_{ab} + \frac{1}{2}(\partial_a\rho)^2 + \frac{1}{8}\rho^2\left(\mathrm{D}^{\hat{C}}_a \boldsymbol{n}\right)^2 + \rho^2\left[(\partial_a \boldsymbol{p})^2 + (\partial_a \boldsymbol{q})^2\right] \\
&+ \frac{\rho^2}{2}\left(n_+\left(\partial_a \hat{\bar{e}}_b\right)^2 + n_-\left(\partial_a \hat{e}_b\right)^2\right) + \frac{1}{2}\rho^2 J_a^2 + \frac{3}{8}\left(1 - n_3^2\right)\rho^4 - \frac{3}{8}\rho^4,
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_{ab} &= (\partial_a J_b - \partial_b J_a) + \frac{1}{2}\boldsymbol{n}\cdot \mathrm{D}^{\hat{C}}_a \boldsymbol{n} \times \mathrm{D}^{\hat{C}}_b \boldsymbol{n} - n_3\left(\partial_a \hat{C}_b - \partial_b \hat{C}_a\right) - 2\rho^2 n_3 H_{ab} \\
J_a &= \frac{\mathrm{i}}{2\rho^2}\left(\psi_1^* \mathrm{D}^{C}_{Aa}\psi_1 - \psi_1 \bar{\mathrm{D}}^{C}_{Aa}\psi_1^* + \psi_2^* \mathrm{D}^{C}_{Aa}\psi_2 - \psi_2 \bar{\mathrm{D}}^{C}_{Aa}\psi_2^*\right)
\end{aligned}
$$

- $\mathrm{U_C}(1) \times \mathrm{U_I}(1)$ - gauge Invariant Fields
- $\boldsymbol{n} \to \mathrm{O}(3)$ - nonlinear σ model
- $(\boldsymbol{p}, \boldsymbol{q}) \to G(4,2) = \frac{\mathrm{O}(4)}{\mathrm{O}(2)\times\mathrm{O}(2)} \sim \mathbb{RP}^2 \times \mathbb{RP}^2$ -nonlinear σ model

Reductions

London Limit $\rho \to \Delta$,

$$\mathcal{L} = \frac{\Delta^2}{8}\left(\mathrm{D}_a^{\hat{C}}\boldsymbol{n}\right)^2 + \frac{3}{8}\Delta^4\left(1 - n_3^2\right) + \frac{1}{16}\left[\boldsymbol{n}\cdot \mathrm{D}_a^{\hat{C}}\boldsymbol{n}\times \mathrm{D}_b^{\hat{C}}\boldsymbol{n} - 2n_3\left(\partial_a \hat{C}_b - \partial_b \hat{C}_a\right)\right]^2.$$

Higgs phenomena $\boldsymbol{n} \to \pm\hat{\boldsymbol{z}}, |\boldsymbol{x}| \to \infty$

- $\boldsymbol{n} \neq \mathrm{cost},\ \rho = \Delta = \mathrm{cost}$ e $J_a = 0$,

$$\mathcal{L} = \frac{\Delta^2}{8}(\partial_a\boldsymbol{n})^2 + \frac{1}{16}\left(\boldsymbol{n}\cdot\partial_a\boldsymbol{n}\times\partial_b\boldsymbol{n} - 4\Delta^2 n_3 H_{ab}\right)^2 - \frac{3}{8}\Delta^4 n_3^2 \quad (H_{4i} = p_i,\ H_{jk} = 2\epsilon_{ijk}q_i)$$

- $\boldsymbol{n} = \hat{\boldsymbol{z}} = \mathrm{cost},\ \rho \neq \mathrm{cost}$ e $J_a \neq 0$,

$$\mathcal{L} = \frac{1}{4}\left(\partial_a J_b - \partial_b J_a - 2\rho^2 H_{ab}\right)^2 + \frac{1}{2}(\partial_a\rho)^2 + \frac{1}{2}\rho^2 J_a^2 - \frac{3}{8}\rho^4 \tag{1a}$$

- $\boldsymbol{n} \neq \mathrm{cost},\ \rho = \Delta = \mathrm{cost}$ e $J_a \neq 0$ (Current States)

$$\mathcal{L} = \frac{1}{4}\left[(\partial_a J_b - \partial_b J_a) + \frac{1}{2}(\boldsymbol{n}\cdot\partial_a\boldsymbol{n}\times\partial_b\boldsymbol{n}) - 2\Delta^2 n_3 H_{ab}\right]^2 + \frac{\Delta^2}{2}J_a^2 + \frac{\Delta^2}{8}(\partial_a\boldsymbol{n})^2 - \frac{3}{8}\Delta^4 n_3^2 \tag{1b}$$

G. Martone, Thesis (Lecce, 2011)

Stereographic form of the Skyrme-Faddeev model

$S^2 \leftrightarrow \mathbb{C} \qquad \boldsymbol{n} = \left(\frac{w+\bar{w}}{w\bar{w}+1}, -\frac{i(w-\bar{w})}{w\bar{w}+1}, \frac{1-w\bar{w}}{w\bar{w}+1} \right) \quad w = \frac{n_1+\imath n_2}{1-n_3}$

$$\mathcal{L}_w = \frac{\sum_{i=0}^{3} g^i \partial_i w \; \partial_i \bar{w}}{8\pi^2 \left(1+w\bar{w}\right)^2} + \lambda \frac{\sum_{i,j=0,i<j}^{3} g^i \; g^j \left(\partial_i w \; \partial_j \bar{w} - \partial_j w \; \partial_i \bar{w}\right)^2}{16\pi^2 \left(1+w\bar{w}\right)^4}.$$

$(g_i) = (-1, 1, 1, 1)$
$U = (w, \bar{w})^T \quad U_i = \partial_i U, \quad U_{i,j} = \partial_i \partial_j U.$

$$\textstyle\sum_{0\le i\le j\le 3} K_{ij}\left[U, U_0, \ldots, U_3\right] U_{ij} - K_0\left[U, U_0, \ldots, U_3\right] = 0$$

$$\begin{aligned} K_{ij} = \quad & g^i \left\{ \delta_{ij} \left[\left(1+\tfrac{1}{2} U^\dagger U\right)^2 \sigma_1 + \tfrac{\lambda}{2} \mathbf{A} \textstyle\sum_l \left(1-\delta_{il}\right) g_l U_l \otimes U_l \right] - \lambda \left(1-\delta_{ij}\right) \mathbf{A} g^j U_i \otimes U_j \right\} \\ K_0 = \quad & \left\{ \left(1+\tfrac{1}{2} U^\dagger U\right) \mathbf{AB} \textstyle\sum_{0\le l\le 3} g_l U_l \otimes U_l - \frac{2\lambda}{1+\frac{1}{2} U^\dagger U} \textstyle\sum_{0\le l<m\le 3} g_l g_m \left[\mathbf{AC} U_l \otimes U_m\right]^2 \right\} U \end{aligned}$$

Lie-point Symmetry Group $\left(\mathbb{E}^4 \rtimes SO(3,1)\right) \otimes SO(3)$

$$\mathbf{t}_i = \partial_i, \qquad \mathbf{r}_{i,j} = x^i \partial_j - g^i g^j x^j \partial_i, \quad (i,j = 0, \cdots, 3)$$

$$\mathbf{w}_0 = -w\partial_w + \bar{w}\partial_{\bar{w}}, \qquad \mathbf{w}_1 = \partial_w + \bar{w}^2 \partial_{\bar{w}}, \quad \mathbf{w}_{-1} = w^2 \partial_w + \partial_{\bar{w}},$$

$$[\mathbf{w}_1, \mathbf{w}_{-1}] = 2\mathbf{w}_0, \quad [\mathbf{w}_0, \mathbf{w}_{\pm 1}] = \pm \mathbf{w}_{\pm 1},$$

$$\tilde{w} = e^{i\eta} \frac{a + i\,w}{1 - i\,\bar{a}\,w} \qquad \eta \in \mathbb{R},\ a \in \mathbb{C}, \qquad \boldsymbol{n} \Leftrightarrow R\boldsymbol{n}$$

$\Leftrightarrow$ Lagrangian Symmetries

$$
\begin{aligned}
P_{i\,i} &= \frac{-2|\partial_i w|^2 + \sum_{j=0}^{3} |\partial_j w|^2}{8\pi^2 \left(1 + w\bar{w}\right)^2} - \lambda \frac{\sum_{j<k} (-1)^{\delta_{i\,j}+\delta_{i\,k}} \left(\partial_j w \, \partial_k \bar{w} - \partial_k w \, \partial_j \bar{w}\right)^2}{16\pi^2 \left(1 + w\bar{w}\right)^4}, \\
P_{i\,j} &= -\frac{Re\left(\partial_i w \partial_j \bar{w}\right)}{8\pi^2 \left(1 + w\bar{w}\right)^2} + \lambda \frac{\sum_{k\neq i,\,j} Im\left(\partial_i w \partial_k \bar{w}\right) Im\left(\partial_j w \partial_k \bar{w}\right)}{4\pi^2 \left(1 + w\bar{w}\right)^4} \\
J_k^{i\,j} &= x_i P_{jk} - x_j P_{jk} \qquad (i,\, j \text{ cyclic}) \\
Q_j &= \frac{Im\left(\bar{w}\partial_i w\right)}{2\pi^2 \ \left(1 + w\bar{w}\right)^2} - \lambda \frac{Im\left(w\left(\partial_j w \sum_{k\neq j} \partial_k \bar{w}^2 - \partial_j \bar{w} \sum_{k\neq j} |\partial_k w|^2\right)\right)}{4\pi^2 \left(1 + w\bar{w}\right)^4}
\end{aligned}
$$

Symmetry Sub-Algebras Classification

$so\,(3,1) \uplus \left(Span\,\{\mathbf{t}_i\} \oplus so\,(3)_{gauge} \right)$

- classify $S_k \subset so\,(3)$: $\quad norS_k = \{\mathbf{v} \in so\,(3) : ad_{\mathbf{v}} S_k \subseteq S_k\}$;
- identify $N_k \subset Span\,\{\mathbf{t}_i\} \oplus so\,(3)_{gauge}$: $\quad NorS_k\,(N_k) \subset N_k$;
- identify *non-splitting sub-algebras* : $\neq S_k \oplus N_k$

- 1-dim : $\mathbf{t}_0$, $\mathbf{r}_{12}$, $\mathbf{t}_3$, $\mathbf{r}_{12} + \alpha \mathbf{w}_0$, $\alpha \in \mathbb{R}$;
- 1-dim non splitting : $\mathbf{r}_{12} \pm \mathbf{t}_3$;
- 2-dim s.a. : $\{\mathbf{t}_0, \mathbf{t}_3\}$, $\{\mathbf{t}_3, \mathbf{t}_1\}$, $\{\mathbf{t}_0, \mathbf{r}_{12}\}$, $\{\mathbf{t}_0, \mathbf{r}_{12} \pm \mathbf{t}_3\}$ and $\{\mathbf{t}_0, \mathbf{r}_{12} + \alpha \mathbf{w}_0\}$ for any $\alpha \in \mathbb{R}$;
- 3-dim : $so\,(3)_{rot}$, $\{\mathbf{t}_0, \mathbf{t}_3, \mathbf{t}_1\}$, $\left\{\mathbf{r}_{12} + \mathbf{w}_0, \mathbf{r}_{23} + \frac{\mathbf{w}_1 + \mathbf{w}_{-1}}{2}, \mathbf{r}_{31} + \frac{\mathbf{w}_1 - \mathbf{w}_{-1}}{2i}\right\}$

1- and 2-dim space reductions $E\,[\boldsymbol{n}] = \infty$

- Plane Waves $w = w_0\, e^{ip_j x_j}$: $\quad \sum_i g^j p_j^2 = 0$
- Monopole $\boldsymbol{n} \rightsquigarrow \pm \hat{\mathbf{r}}$

Hedgehog Solutions

$$\boldsymbol{v} = \imath\left(x\partial_y - y\partial_x\right) + \alpha\left(w\partial_w - \bar{w}\partial_{\bar{w}}\right) \Rightarrow w = e^{\imath n\varphi}\left(\cot[\theta] + \imath\cot[\chi\left(r\right)]\csc[\theta]\right)$$

$$\begin{aligned} \boldsymbol{n}\cdot\boldsymbol{\sigma} &= U\left(\boldsymbol{n}_\infty\cdot\boldsymbol{\sigma}\right)U^\dagger \\ U = \exp\left[\mathrm{i}\chi\left(r\right)\boldsymbol{\nu}\left(\vartheta,\,\varphi\right)\cdot\boldsymbol{\sigma}\right] &= \cos\chi\left(r\right)I + \mathrm{i}\sin\chi\left(r\right)\boldsymbol{\nu}\left(\vartheta,\,\varphi\right)\cdot\boldsymbol{\sigma} \end{aligned}$$

$$\boldsymbol{\nu}\left(\vartheta,\,\varphi\right) = \left(\sin\left(m\vartheta\right)\cos\left(n\varphi\right),\sin\left(m\vartheta\right)\sin\left(n\varphi\right),\cos\left(m\vartheta\right)\right) : \mathbb{S}^2 \to \mathbb{S}^2 \; deg\left(\boldsymbol{\nu}\right) = m\,n$$

$$E\left[\chi\right] = \frac{1}{3\pi}\int_0^\infty\left\{r^2\chi'^2 + 2\sin^2\chi\left(\lambda\chi'^2 + 1\right) + \lambda\frac{\sin^4\chi}{r^2}\right\}dr$$

$$\left(r^2 + 2\lambda\sin^2\chi\right)\chi'' + 2r\chi' + \sin 2\chi\left(\lambda\chi'^2 - 1 - \lambda\frac{\sin^2\chi}{r^2}\right) = 0$$

$$\chi\left(0\right) = \pi \text{ and } \chi\left(\infty\right) = 0$$

Hedgehog Solutions

$$g\left(r\right)=\sin\frac{\chi\left(r\right)}{2}\qquad\left(\lambda=1\right)$$

$$\left(8g^4-8g^2-r^2\right)\left(g^2-1\right)g''+g\left[8g^2\left(g^2-2\right)+r^2+8\right]g'^2$$
$$-2r\left(g^2-1\right)g'-\frac{2g\left(2g^2-1\right)\left(g^2-1\right)^2\left(4g^4-4g^2-r^2\right)}{r^2}=0,$$

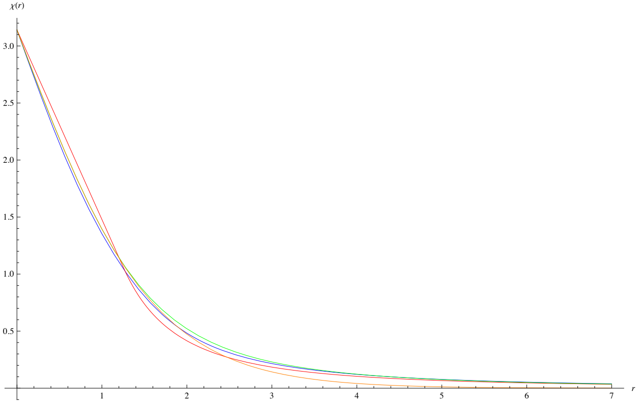
NO Painlevé property

Approximated solutions by rational f.

$$g_{rat}\left(r\right)=\frac{1+a_1r+a_2r^2}{1+a_1r+b_2r^2+b_3r^3+b_4r^4},$$

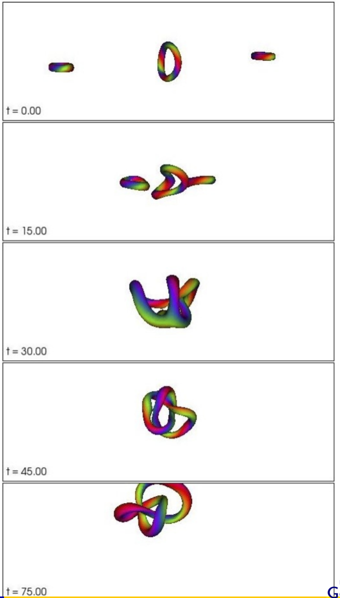
$$a_1=0.216,\quad a_2=0.230,\quad b_2=0.752,\quad b_3=-0.018,\quad b_4=0.302,$$

Hedgehog Profile



Blu : numerical solution. Green: $\chi_{rat} = 2 \arcsin g_{rat}$. Red: test χ_p-function. Orange: Atiyah - Manton test function $\frac{|E[\chi_{num}] - E[\chi_{rat}|]}{|E[\chi_{num}]|} \approx 10^{-3}$

Collisions of Vortices (*Hietarinta et al '11*)



Rational Maps Ansatz

(Manton) $\quad \omega \in K \subset SO(3)$

$\mathbb{R}^3 \to \mathbb{S}^2 \times \mathbb{R}^+$ $z \to \omega_S(z) = \frac{\alpha z + \beta}{-\bar{\beta} z + \bar{\alpha}}, \quad |\alpha|^2 + |\beta|^2 = 1$

w-space $w \to \omega_T(w) = \frac{\gamma w + \delta}{-\bar{\delta} w + \bar{\gamma}}, \quad |\gamma|^2 + |\delta|^2 = 1$

symmetric map: $w(\omega_S(z)) = \omega_T(w(z)) \quad \forall \omega \in K$

IRREP $SO(3)$ subgroups (Platonic symm) $\Leftrightarrow$ Klein Polynomials

$$R_C = z, R_D = z^2, R_T = \frac{z^3 - \sqrt{3}\imath z}{\sqrt{3}\imath z^2 - 1}, R_O = \frac{z^4 + 2\sqrt{3}\imath z^2 + 1}{z^4 - 2\sqrt{3}\imath z^2 + 1},$$

$$R_Y = \frac{z^7 - z^5 - 7z^2 - 1}{z^7 + z^5 - 7z^2 + 1}$$

F. Klein, Lectures on the Icosahedron, (London, Kegan Paul, 1913)

Rational Maps Ansatz

$\boldsymbol{\nu}_R = \frac{1}{1+|R|^2}\left(R+\bar{R}, -\imath\left(R-\bar{R}\right), 1-|R|^2\right)$ $U_R = \exp\left[\mathrm{i}\chi\left(r\right)\boldsymbol{\nu}_R \cdot \boldsymbol{\sigma}\right]$

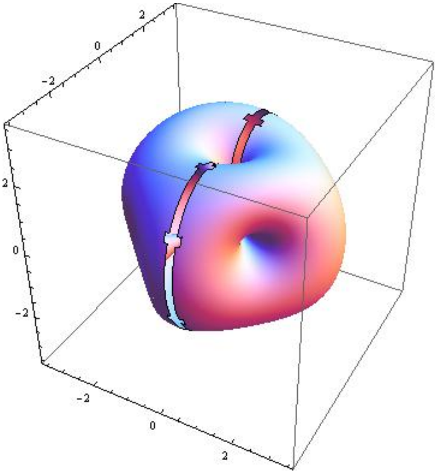
$$w\left(r, z, \bar{z}\right) = \frac{(1-|R|^2) + i(1+|R|^2)\cot\chi\left(r\right)}{2\bar{R}}.$$

$$E\left[w, \bar{w}\right] = \int \left\{ \frac{|\partial_i w|^2}{8\pi^2(1+|w|^2)^2} - \lambda \frac{\left(\partial_i w\, \partial_j \bar{w} - \partial_j w\, \partial_i \bar{w}\right)^2}{32\pi^2(1+|w|^2)^4} \right\} \frac{2idzd\bar{z}}{(1+|z|^2)^2}\, r^2 dr.$$

$$E\left[\chi\right] = \frac{1}{3\pi}\int_0^\infty \left\{ \mathcal{I} r^2 \chi'^2 + 2\sin^2\chi\left(\lambda \mathcal{B}_1 \chi'^2 + \mathcal{B}_2\right) + \lambda \mathcal{J} \frac{\sin^4\chi}{r^2} \right\} dr,$$

$\mathcal{I} = \frac{3}{2\pi}\int \frac{|R|^2}{(1+|R|^2)^2}\frac{2idzd\bar{z}}{(1+|z|^2)^2}$ $\mathcal{J} = \frac{3}{4\pi}\int \left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{dR}{dz}\right|\right)^4 \left(\frac{1-|R|^2}{1+|R|^2}\right)^2 \frac{2idzd\bar{z}}{(1+|z|^2)^2}$
$\mathcal{B}_1 = \mathcal{B}_2 = N$

Rational map $R(z)$	Degree N	Coefficient $\mathcal{I}$	Coefficient $\mathcal{J}$
z	1	1	1
z^2	2	0.644	3.956
$\frac{z^3-\sqrt{3}iz}{\sqrt{3}iz^2-1}$	3	1	13.577
$\frac{z^4+2\sqrt{3}iz^2+1}{z^4-2\sqrt{3}iz^2+1}$	4	1.172	25.709
$\frac{z^7-7z^5-7z^2-1}{z^7+7z^5-7z^2+1}$	7	1	60.868



$$R_T = \frac{z^3 - \sqrt{3}\imath z}{\sqrt{3}\imath z^2 - 1}$$

Polar form of the Skyrme - Faddeev model

Polar representation

$$\phi = (\sin w \cos u, \sin w \sin u, \cos w)\,, \tag{2}$$

$$\mathcal{L}_p = \frac{1}{32\pi^2}\,\left\{ w_\mu w^\mu + \sin^2 w \left[u_\mu u^\mu - \frac{\lambda}{2}\left(w_\mu w^\mu u_\nu u^\nu - w_\mu w_\nu u^\mu u^\nu \right)\right]\right\}, \tag{3}$$

Euler - Lagrange Equations

$$\begin{gathered} \partial_\mu w^\mu = \tfrac{1}{2}\sin(2w) u_\nu u^\nu + \tfrac{\lambda}{2}\sin w\ u_\nu\, \partial_\mu[\sin w (w^\mu u^\nu - w^\nu u^\mu)], \\ w_\mu u^\mu \sin(2w) + \sin^2 w[\partial_\mu u^\mu + \tfrac{\lambda}{2} w_\nu \partial_\mu (u^\mu w^\nu - u^\nu w^\mu)] = 0. \end{gathered} \tag{4}$$

The d'Alembert-homogeneous Eikonal reduction: $w = const$

$$\partial_\mu u^\mu = 0, \qquad u_\nu u^\nu = 0,$$

$$G\left(u, A_\mu(u)\, x^\mu, B_\mu(u)\, x^\mu\right) = 0, \qquad A_\mu A^\mu = B_\mu B^\mu = A_\mu B^\mu = 0,$$

with G, A_μ and B_μ arbitrary real regular functions.

Cieciura G and Grundland A M 1984 *J. Math. Phys.* **2** 3460-3469
Collins C B 1983 *J. Math. Phys.* **24** 22
Fushchich V I, Zhdanov R Z and Revenko I V 1991 *Ukr. Mat. Z.* **43** 1471-1487; Zhdanov R Z, Revenko I V and Fushchich V I 1995 *J. Math. Phys.* **36** 7109

Orthogonality reduction

By imposing

$$w_\mu u^\mu = 0, \qquad u_\nu u^\nu = \alpha \quad (\alpha = \text{ constant } \in \mathbb{R}) .$$

the system reduces to the equations

$$\begin{aligned} \partial_\mu u^\mu = 0, &\qquad u_\nu u^\nu = \alpha, \\ w_\mu u^\mu = 0, &\qquad \partial_\mu w^\mu = \frac{\alpha}{2} \frac{\sin(2w)}{1 - \frac{\lambda\alpha}{2}\sin^2 w}\left(1 + \frac{\lambda}{2} w^\mu w_\mu\right), \end{aligned}$$

which are highly nonlinear for the w field.

General solution of the d'Alembert-Eikonal system

$$\begin{aligned} u &= A_\mu(\tau)\, x^\mu + R_1(\tau), \\ & \quad B_\mu(\tau)\, x^\mu + R_2(\tau) = 0, \\ A_\mu A^\mu &= \alpha, \quad A_\mu B^\mu = A'_\mu B^\mu = B_\mu B^\mu = 0, \end{aligned}$$

Then, for $\alpha = -\eta^2$, the general solution is

$$u = x_k A_k(\tau) + A_0(\tau), \quad t = x_k B_k(\tau) + B_0(\tau),$$
$$A_1 = \eta \cos(f(\tau)) \sin(g(\tau)), A_2 = \eta \sin(f(\tau)) \sin(g(\tau)), A_3 = \eta \cos(g(\tau)),$$

being $f(\tau)$ and $g(\tau)$ arbitrary functions.

The reduced Skyrme–Faddeev system

By setting to zero the coefficients of all functions of $w \Rightarrow$ quasilinear system in $(u^\mu,\, w^\mu)$

$$\partial_\mu w^\mu = 0, \qquad w_\mu w^\mu = -\epsilon^2, \qquad u_\mu w^\mu = 0, \tag{5}$$
$$\text{d'Alembert-Eikonal} \quad \left(u \to w, \alpha \to -\epsilon^2\right) \quad \text{orthogonality condition}$$

$$u_\nu \partial_\mu (w^\mu u^\nu - w^\nu u^\mu) = 0, \quad \epsilon^2 \partial_\mu u^\mu + w_\nu \partial_\mu (u^\mu w^\nu - u^\nu w^\mu) = 0,$$
$$\Updownarrow\, a_\mu w^\mu = 0 \quad \text{with} \quad a = u^\nu u_\nu \qquad\qquad \text{identity} \ \left(\epsilon^2 = \tfrac{2}{\lambda}\right) \tag{6}$$

Compatibility condition for d'Alembert-Eikonal eq. : the Monge-Ampére

$$\mathrm{Det}\,[w_{ij}] = 0,$$

Compatibility condition for the u-orthogonality eq.s

$$
\begin{gathered}
(w_s^2-\epsilon^2)u_m u_k w_{km} + (u_k w_k)^2 w_{mm} = 2u_s w_s u_m w_k w_{km},\\
4u_k w_k u_s w_{sp}(w_m w_{pm} - w_p w_{mm}) + 2(u_s w_m w_{sm})^2 + \\
(u_s w_s)^2(w_{mm}w_{pp} - w_{pm}^2) = 2(w_p^2-\epsilon^2)(u_s w_{sm})^2.
\end{gathered}
$$

Search for

$$
u_0 = A\left(w_i, w_{ij}\right)\, u_1, \quad u_2 = B\left(w_i, w_{ij}\right)\, u_1, \quad u_3 = C\left(w_i, w_{ij}\right)\, u_1,
$$

$$
\begin{array}{lcl}
(1{+}2)\text{-dim} & \Rightarrow & u = F\left[w_1, w_2\right], \quad a \equiv 0 \quad \forall w \\
(1{+}3)\text{-dim} & \Rightarrow & u = F\left[w_1, w_2, w_3\right]
\end{array}
$$

$$
\begin{gathered}
(x_m B_m'(\tau) + B_0'(\tau))d\tau = dt - B_k(\tau)dx_k, \quad X_k = x_k,\\
[X_m(B_m'(\tau)A_p(\tau) - A_m'(\tau)B_p(\tau)) + B_0'(\tau)A_p(\tau) - A_0'(\tau)B_p(\tau)]u_{X_p} = 0
\end{gathered}
$$

Phase - pseudo-phase Solutions

$$w = \Theta\left[\theta\right], \;\; u = \Phi\left[\theta\right] + \tilde{\theta}, \;\text{where}\;\; \theta = \alpha_\mu x^\mu, \tilde{\theta} = \beta_\mu x^\mu$$

A 3-parametric family of equations

$$\left[2B_3 - \frac{\lambda}{4}\mathcal{B}\sin^2\Theta\right]\Theta_{\theta\theta} = \sin 2\Theta\left(\frac{\lambda}{8}\mathcal{B}\;\Theta_\theta^2 + B_3\Phi_\theta^2 + B_2\Phi_\theta + B_1\right)$$
$$2B_3\sin^2\Theta\;\Phi_{\theta\theta} + \Theta_\theta\sin 2\Theta\left(2B_3\Phi_\theta + B_2\right) = 0,$$

where $B_1 = -\beta_\mu\beta^\mu$, $B_2 = -2\alpha_\mu\beta^\mu$ $B_3 = -\alpha_\mu\alpha^\mu$ and $\mathcal{B} = B_2^2 - 4B_1B_3$.

Conservation laws

$$\begin{aligned}\mathcal{E}^0 &= B_3\alpha_0\Theta_\theta^2 + \sin^2\Theta\left[2\boldsymbol{\alpha}\cdot\boldsymbol{\beta}\,\beta_0 + \left(B_1 - 2\boldsymbol{\beta}^2\right)\alpha_0\right.\\ &\quad \left. + B_3\left(2\beta_0 + \alpha_0\Phi_\theta\right)\Phi_\theta - \frac{\lambda\mathcal{B}}{8}\alpha_0\Theta_\theta^2\right],\\ \mathcal{E}^i &= B_3\alpha_i\Theta_\theta^2 + \sin^2\Theta\left[B_2\beta_i - B_1\alpha_i + B_3\left(2\beta_i + \alpha_i\Phi_\theta\right)\Phi_\theta - \frac{\lambda\mathcal{B}}{8}\alpha_i\Theta_\theta^2\right].\end{aligned}$$

$B_3 \neq 0$ and substitution

$$\Theta = \arcsin\sqrt{\psi},$$

$$\psi_\theta^2 = \frac{64(\psi-1)\left(\psi-A_1\right)\left(\psi-A_2\right)}{\lambda^2\mathcal{B}\psi_1\left(\psi_1-\psi\right)}.$$

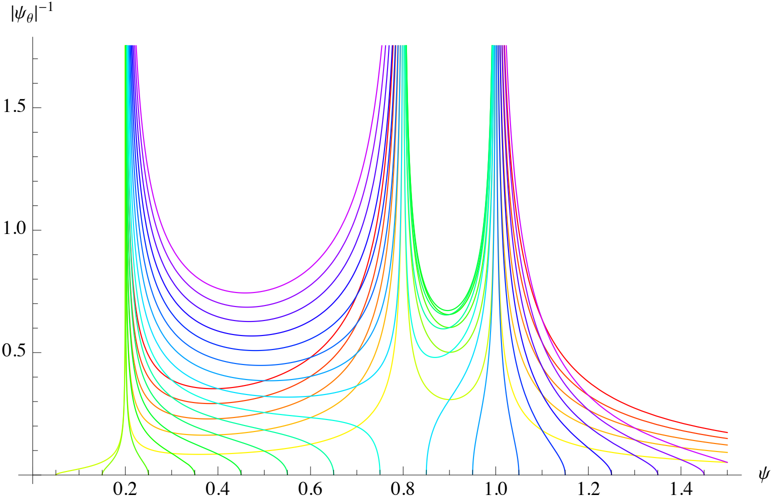
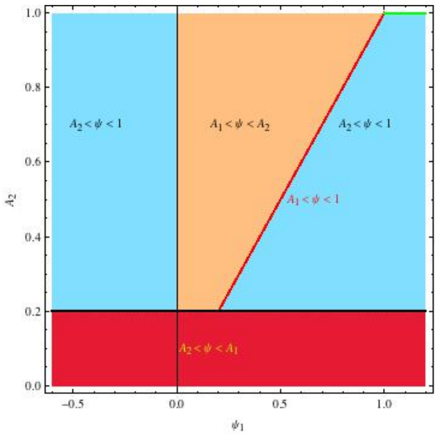


Figura: The graphic for the inverse square root of ψ_θ for the family of parameters $\mathcal{B} = 1, A_1 = .1, A_2 = .8$ and $-.45 \leq \psi_1 \leq 1.55$ with steps of 0.1. Only one bounded periodic solutions exists for any set of parameters.

Parametric form solutions

$$
\begin{aligned}
\theta(\psi) &= \theta_0 + \frac{1}{4}\sqrt{\frac{\mathcal{B}\lambda^2\psi_1(\psi_1 - A_1)^2}{(A_1-1)(A_2-\psi_1)}}\,\Pi\left[\frac{A_1-A_2}{\psi_1-A_2}; Z|\frac{(\psi_1-1)(A_1-A_2)}{(A_1-1)(\psi_1-A_2)}\right],\\
\psi &= -\frac{A_2\psi_1\sin^2 Z + A_1\left(\psi_1\cos^2 Z - A_2\right)}{A_1\sin^2 Z + A_2\cos^2 Z + \psi_1}
\end{aligned}
$$

$$
\begin{aligned}
\Phi &= -\frac{B_2U_2}{2B_3}\left[\int\frac{d\theta}{\psi(\theta)} + \theta\right] + \Phi_0 = \\
&- \frac{s_1}{2\psi_1}\left[\sqrt{\frac{2\psi_1(A_1-\psi_1)^2(B_1\lambda\psi_1+2)}{(A_1-1)(A_2-\psi_1)}}\,\Pi\left(\frac{A_2-A_1}{A_2-\psi_1}; Z\left|\frac{(A_1-A_2)(\psi_1-1)}{(A_1-1)(\psi_1-A_2)}\right.\right)\right.\\
&\left. +2s_2\sqrt{\frac{A_2\psi_1(A_1-\psi_1)^2}{A_1(A_1-1)(A_2-\psi_1)}}\,\Pi\left(\frac{(A_1-A_2)\psi_1}{A_1(\psi_1-A_2)}; Z\left|\frac{(A_1-A_2)(\psi_1-1)}{(A_1-1)(\psi_1-A_2)}\right.\right)\right],
\end{aligned}
$$



(ψ_1, A_2) phase plane of the amplitudes of the periodc ψ function

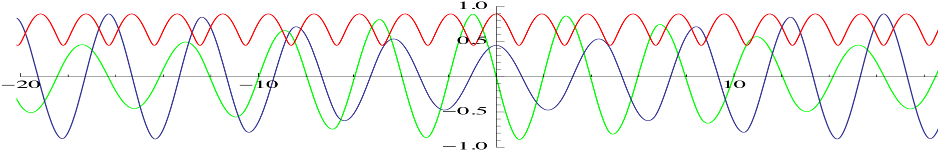


Figura: The graphic for the ϕ_1 (green), ϕ_2 (blue) and ϕ_3 () as function of x^3 for a choice of the parameters
$A_1 = 0.2, A_2 = 0.8, \psi_1 = 0.9, \mathcal{B} = 1, \lambda = 1, B_1 = 1, s_1 = -1, s_2 = -1$.
Accordingly, the wave vectors for the phase and pseudo-phase have been chosen to be $\alpha_\mu = (0, 0, 0, 0.33541)$ and $\beta_\mu = (1.49638, 1, 0, -1.49638)$, respectively

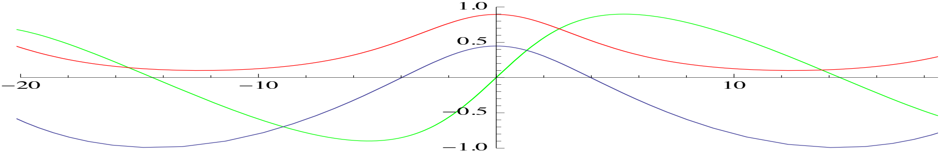
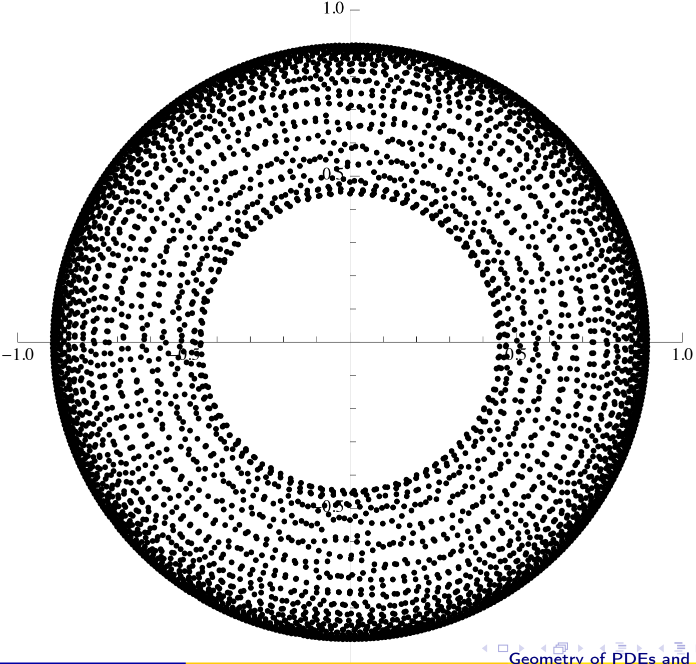
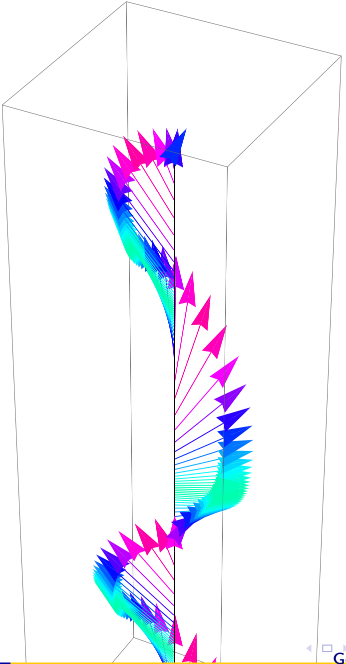


Figura: The graphic for the ϕ_1 (green), ϕ_2(blue) and ϕ_3 () for a choice of the parameters
$A_1 = 0.2, A_2 = 0.99, \psi_1 = 20.01, \mathcal{B} = 1, \lambda = 1, B_1 = 1, s_1 = -1, s_2 = -1$. The wave vectors are $\alpha_\mu = (0, 0, 0, -1.58153)$ and $\beta_\mu = (-1.04879, 1, 0, 1.04879)$, respectively


1.0
0.5
−1.0
−0.5
0.5
1.0
−0.5



The Whitham averaging method

$$\hat{\mathcal{L}}_p = \sin^2(\Theta)\left(-\frac{1}{2}\lambda\left(\frac{B_2^2}{4} - B_1 B_3\right)\Theta_\theta^2 + B_3\Phi_\theta^2 + B_2\Phi_\theta + B_1\right) + B_3\Theta_\theta^2,$$

Averaged constrained Lagrangian on a period

$$L \equiv \frac{1}{2\pi}\oint \hat{\mathcal{L}}_p \, d\theta, \quad \oint d\theta = 2\pi, \quad < \Phi_\theta >= \oint \Phi d\theta = 2\pi m,$$

$$L = \left(B_1 - \frac{B_2^2}{4B_3}\right)\left(A_1 + A_2 + W\sqrt{\frac{\lambda}{2}B_3}\right) + \frac{B_2 + 2mB_3}{2B_3}\sqrt{A_1 A_2 (B_2^2 - 4B_1 B_3)},$$

where

$$W = \frac{1}{2\pi}\oint \sqrt{\frac{(\psi - A_1)(\psi - A_2)(\psi - \psi_1)}{1-\psi}}\frac{d\psi}{\psi}.$$

$L_{A_1} = 0$ and $L_{A_2} = 0$
$\omega = -\theta_{X^0}, k_i = \theta_{X^i}$ and $\gamma = -\tilde{\theta}_{X^0}, \beta_i = \tilde{\theta}_{X^i}$, where X^0, X^1, X^2, X^3 are the so called "slow" variables in comparison with "fast" variables x^0, x^1, x^2, x^3

$$\partial_0 L_\omega = \partial_i L_{k^i}, \quad \partial_0 L_\gamma = \partial_i L_{\beta^i}, \tag{7}$$

with the compatibility conditions

$$\begin{array}{rcl} \partial_0 k^1 + \partial_i \omega & = & 0, \quad , \quad \partial_j k^i = \partial_i k^j \quad i \neq j, \\ \partial_0 \beta^i + \partial_i \gamma & = & 0, \quad , \quad \partial_j \beta^i = \partial_i \beta^j \quad i \neq j. \end{array} \tag{8}$$

Conclusions and open problems

- Skyrme-Faddeev model is relevant in Condensed Matter and pure Yang-Mills theory in infrared limit
- Localized perturbations are Knotted Vortices stabilized by the Hopf index
- Approximate solutions can be found in the axisymmetric setting and/or in the rational map ansatz
- Domain -wall solutions are described by the d'Alembert-Eikonal system and its generalizations
- Periodic Solutions exists in terms of elliptic functions (Indication of integrable sub-sectors?)
- Witham averaging method can be applied, but challenging
- Higher symmetries (if any) are still unknown
- Reduction/ modification to integrable systems is unknown (not even in 2D)
- Interaction among hopfions is under considerations by numericals and by lattice toroidal moment models (Protogenov,Verbus)