Self-consistent sources for integrable equations via deformations of binary Darboux transformations

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Introduction

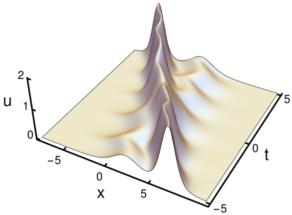
Appearance of soliton equations with self-consistent sources:

- Mel'nikov 1983-92, ...
- *Mathematically:*
    - via a multiscaling limit of familiar integrable systems (Zakharov & Kuznetsov 1986, ...)
    - via a symmetry constraint imposed on a higher than two-dimensional integrable system (Konopelchenko, Sidorenko, Strampp 1991, Oevel 1993, ...)
- *Physically:* often a certain approximation of nonlinear interaction of waves with long and short wave length.

Example 1: KdV with self-consistent sources

$$4\,u_t - u_{xxx} - 3\,(u^2)_x = (r^{\intercal}\,r)_x \qquad\qquad r_{xx} = P^2\,r - u\,r$$

Without the source on the rhs of the KdV equation, the last equation is half of its Lax pair. Somehow an equation $r_t = \ldots$ seems to be missing (e.g., in order to address integrability).



Plot of u for a 2-soliton solution of the above scalar KdV equation with self-consistent sources. Here an arbitrary function of t that is present in the solution has been chosen as $\sin(3t)$. There are lots of possibilities ...

Example 2: (2+1)-dimensional Yajima-Oikawa system

Yajima-Oikawa system

$$u_t = (r^\intercal r)_x \qquad \mathrm{i}\, r_t = r_{xx} - u\, r$$

Integrable three-dimensional generalization (Oikawa, Okamura, Funakoshi '89):

$$u_t = (r^\intercal r)_x \qquad \mathrm{i}\, (r_t + r_y) = r_{xx} - u\, r$$

This looks like a reasonable evolution system ..., but via the transformation $\xi = x + y - t$, $\tau = t - y$, $\eta = y$, we get

$$u_\tau = u_\xi + (r^\intercal r)_\xi \qquad \mathrm{i}\, r_\eta = r_{\xi\xi} - u\, r$$

(Grimshaw '77, Mel'nikov '83). Again, somehow an equation $r_\tau = \dots$ is missing.

We will present a more coherent picture in the sequel.

Guiding example: potential KP equation

$$\left(4\,\phi_{0,t} - \phi_{0,xxx} - 6\,(\phi_{0,x})^2\right)_x$$
$$-3\,\phi_{0,yy} + 6\,[\phi_{0,x}, \phi_{0,y}] = 0$$

Binary Darboux transformation for pKP

Associated linear system and its adjoint:

$$\theta_y = \theta_{xx} + 2\,\phi_{0,x}\,\theta \qquad \theta_t = \theta_{xxx} + 3\,\phi_{0,x}\,\theta_x + \frac{3}{2}\,(\phi_{0,y} + \phi_{0,xx})\,\theta$$

$$\eta_y = -\eta_{xx} - 2\,\eta\,\phi_{0,x} \qquad \eta_t = \eta_{xxx} + 3\,\eta_x\,\phi_{0,x} - \frac{3}{2}\,\eta\,(\phi_{0,y} - \phi_{0,xx})$$

These equations imply the compatibility of the system

$$\Omega_x = -\eta\,\theta \qquad \Omega_y = -\eta\,\theta_x + \eta_x\,\theta$$

$$\Omega_t = -\eta\,\theta_{xx} + \eta_x\,\theta_x - \eta_{xx}\,\theta - 3\,\eta\,\phi_{0,x}\,\theta$$

hence the existence of a "potential" Ω. Then it follows that

$$\phi = \phi_0 - \theta\,\Omega^{-1}\,\eta$$

is a new solution of the pKP equation, and

$$q = \theta\,\Omega^{-1} \qquad r = \Omega^{-1}\,\eta$$

solve the linear equations with ϕ instead of ϕ_0. This is an essential step in the binary Darboux transformation for the pKP equation.

Remark

If ϕ is an $m \times m$ matrix, then
θ, q have matrix size $m \times n$,
η, r have size $n \times m$,
Ω and ω are $n \times n$ matrices.

Hence we consider a *vectorial* binary Darboux transformation, so there is no need to iterate it.

Deformation of the potential Ω

$$
\begin{aligned}
\Omega_x &= -\eta\,\theta + c_1\,, \\
\Omega_y &= -\eta\,\theta_x + \eta_x\,\theta + c_2\,, \\
\Omega_t &= -\eta\,\theta_{xx} + \eta_x\,\theta_x - \eta_{xx}\,\theta - 3\,\eta\,\phi_{0,x}\,\theta + c_3
\end{aligned}
$$

with (matrix) functions c_i, $i = 1, 2, 3$. Consistency requires that

$$
c_1 = \omega_x \qquad c_2 = \omega_y \qquad c_3 = \omega_t
$$

with a potential ω. Hence this deformation actually amounts to the substitution

$$
\Omega \mapsto \Omega - \omega
$$

in the previous equations. Then ϕ, q, r no longer satisfy the pKP equation and the linear systems! Instead we find ...

$$
\begin{aligned}
q_y - q_{xx} &= 2\,\phi_x\, q + (-q\,\omega_y + q\,\omega_{xx} + 2\,q_x\,\omega_x)\;\Omega^{-1}\,, \\
q_t - q_{xxx} &= 3\,\phi_x q_x + \frac{3}{2}\,(\phi_y + \phi_{xx})\; q + \frac{3}{2}\,(-q\,\omega_y + q\,\omega_{xx} + 2\,q_x\,\omega_x)\; r\, q \\
&\quad -(q\,\omega_t - q\,\omega_{xxx} - 3\,(q_x\,\omega_x)_x - 3\,\phi_x\,q\,\omega_x)\;\Omega^{-1} \\
r_y + r_{xx} &= -2\,r\,\phi_x - \Omega^{-1}\,(\omega_y\,r + \omega_{xx}\,r + 2\,\omega_x\,r_x)\,, \\
r_t - r_{xxx} &= 3\,r_x\,\phi_x - \frac{3}{2}\,r\;(\phi_y - \phi_{xx}) + \frac{3}{2}\,r\,q\;(\omega_y\,r + \omega_{xx}\,r + 2\,\omega_x\,r_x) \\
&\quad -\Omega^{-1}\;(\omega_t\,r - \omega_{xxx}\,r - 3\,(\omega_x\,r_x)_x - 3\,\omega_x\,r\,\phi_x)
\end{aligned}
$$

and the extended pKP equation

$$
\begin{aligned}
&\left(4\,\phi_t - \phi_{xxx} - 6\,(\phi_x)^2\right)_x - 3\,\phi_{yy} + 6\,[\phi_x, \phi_y] \\
&\quad = (4\,q\,\omega_t\,r - 6\,q\,\omega_x\,r_y - 6\,q\,\omega_x\,r_{xx} + 6\,q\,\omega_y\,r_x - 18\,q\,\omega_{xx}\,r_x \\
&\qquad -6\,q\,\omega_{xy}\,r - 10\,q\,\omega_{xxx}\,r)_x + (6\,q\,\omega_x\,r_x - 3\,q\,\omega_y\,r + 9\,q\,\omega_{xx}\,r)_y \\
&\qquad + (6\,q\,\omega_x\,r_x - 3\,q\,\omega_y\,r + 9\,q\,\omega_{xx}\,r)_{xx}
\end{aligned}
$$

Now we look for choices of ω, such that Ω gets eliminated in some of the equations for q and r ! This requires $\omega_x = 0$ and either ...

1. $\omega = \omega(t)$

$$\left(4\,\phi_t - \phi_{xxx} - 6\,(\phi_x)^2\right)_x - 3\,\phi_{yy} + 6\,[\phi_x, \phi_y] = 4\,\left(q\,\omega_t\,r\right)_x$$
$$q_y - q_{xx} = 2\,\phi_x\,q \qquad r_y + r_{xx} = -2\,r\,\phi_x$$

Via $\tilde{q} = q\,\mathcal{Q}$ and $\tilde{r} = \mathcal{R}\,r$, with a suitable choice of $\mathcal{Q}(t)$ and $\mathcal{R}(t)$, we can absorb ω_t :

$$\left(4\,\phi_t - \phi_{xxx} - 6\,(\phi_x)^2\right)_x - 3\,\phi_{yy} + 6\,[\phi_x, \phi_y] = (\tilde{q}\,\tilde{r})_x$$
$$\tilde{q}_y - \tilde{q}_{xx} = 2\,\phi_x\,\tilde{q} \qquad \tilde{r}_y + \tilde{r}_{xx} = -2\,\tilde{r}\,\phi_x$$

Mel'nikov, ...

2. $\omega = \omega(y)$

$$\begin{aligned} & \left(4\,\phi_t - \phi_{xxx} - 6\,(\phi_x)^2\right)_x - 3\,\phi_{yy} + 6\,[\phi_x, \phi_y] \\ = \; & 3\,\left(q\,\omega_y\,r_x - q_x\,\omega_y\,r\right)_x - 3\,\left(q\,\omega_y\,r\right)_y \end{aligned}$$

$$q_t - q_{xxx} = 3\,\phi_x\,q_x + \frac{3}{2}\,(\phi_y + \phi_{xx})\,q - \frac{3}{2}\,q\,\omega_y\,r\,q$$
$$r_t - r_{xxx} = 3\,r_x\,\phi_x - \frac{3}{2}\,r\,(\phi_y - \phi_{xx}) + \frac{3}{2}\,r\,q\,\omega_y\,r$$

Again, ω_y can be absorbed.

Remarks

The above procedure provides us with a **hetero binary Darboux transformation** from the pKP equation and its associated linear system to any of the pKP systems with self-consistent sources.

The rank of ω_t, respectively ω_y, determines the number of sources.

The method solves the extended (or deformed) pKP system, for *any* given (matrix) function $\omega(t)$, resp. $\omega(y)$.

Via the step to $\tilde{q}$ and $\tilde{r}$, this is then turned into a solution of the extended pKP system, where ω is absorbed.

As a consequence, the scs-extensions of pKP admit solutions depending on *arbitrary* functions of a single variable !

Exact pKP solutions in case of vanishing seed

If $\phi_0 = 0$, special solutions of linear and adjoint linear system are

$$\theta = A\, e^{\vartheta(P)}\, B \qquad \eta = C\, e^{-\vartheta(Q)}\, D \qquad \vartheta(P) = P\, x + P^2\, y + P^3\, t$$

with constant matrices A, B, C, D, P and Q (of appropriate size).

$$\Longrightarrow \quad \Omega = \omega - C\, e^{-\vartheta(Q)}\, X\, e^{\vartheta(P)}\, B$$

with a constant matrix X that satisfies the Sylvester equation

$$X\, P - Q\, X = D\, A$$

Now

$$\phi = \phi_0 - \theta\, \Omega^{-1}\, \eta \qquad q = \theta\, \Omega^{-1} \qquad r = \Omega^{-1}\, \eta$$

yields explicit solutions of the extended matrix pKP equations with self-consistent sources (and with $\omega(t)$, respectively $\omega(y)$).

In the framework of **bidifferential calculus**, we can abstract the underlying structure from the specific example (here pKP) and then obtain corresponding self-consistent source extensions of quite a number of other integrable equations.

Bidifferential calculus

A *graded associative algebra* is an associative algebra $\boldsymbol{\Omega} = \bigoplus_{r \geq 0} \boldsymbol{\Omega}^r$ over $\mathbb{C}$, where $\mathcal{A} := \boldsymbol{\Omega}^0$ is an associative algebra over $\mathbb{C}$ and $\boldsymbol{\Omega}^r$, $r \geq 1$, are $\mathcal{A}$-bimodules such that $\boldsymbol{\Omega}^r \, \boldsymbol{\Omega}^s \subseteq \boldsymbol{\Omega}^{r+s}$.

A *bidifferential calculus* is a unital graded associative algebra $\boldsymbol{\Omega}$, supplied with two ($\mathbb{C}$-linear) graded derivations $\mathrm{d}, \bar{\mathrm{d}} : \boldsymbol{\Omega} \to \boldsymbol{\Omega}$ of degree one (hence $\mathrm{d}\boldsymbol{\Omega}^r \subseteq \boldsymbol{\Omega}^{r+1}$, $\bar{\mathrm{d}}\boldsymbol{\Omega}^r \subseteq \boldsymbol{\Omega}^{r+1}$), and such that

$$\mathrm{d}^2 = \bar{\mathrm{d}}^2 = \mathrm{d}\bar{\mathrm{d}} + \bar{\mathrm{d}}\mathrm{d} = 0$$

Several integrable equations can be expressed either as

$$\mathrm{d}\,\bar{\mathrm{d}}\,\phi_0 + \mathrm{d}\phi_0 \; \mathrm{d}\phi_0 = 0$$

with $\phi_0 \in \mathrm{Mat}(m, m, \mathcal{A})$, or as

$$\mathrm{d}\,[(\bar{\mathrm{d}} g_0)\, g_0^{-1}] = 0$$

The two equations are related by the *Miura equation*

$$\bar{\mathrm{d}} g_0 = (\mathrm{d}\phi_0)\, g_0$$

which has both equations as integrability conditions.

Linear and adjoint linear system

A linear system and an adjoint linear system for the above equation for ϕ_0 is given by

$$\bar{\mathrm{d}}\theta = (\mathrm{d}\phi_0)\,\theta + (\mathrm{d}\theta)\,\Delta$$

respectively

$$\bar{\mathrm{d}}\eta = -\eta\,\mathrm{d}\phi_0 + \Gamma\,\mathrm{d}\eta$$

where $\theta \in \mathrm{Mat}(m, n, \mathcal{A})$, $\eta \in \mathrm{Mat}(n, m, \mathcal{A})$, $\Delta, \Gamma \in \mathrm{Mat}(n, n, \mathcal{A})$. They have to satisfy

$$\bar{\mathrm{d}}\Delta = (\mathrm{d}\Delta)\,\Delta \qquad \bar{\mathrm{d}}\Gamma = \Gamma\,\mathrm{d}\Gamma$$

as a consequence of the properties of d and $\bar{\mathrm{d}}$.

Binary Darboux transformation

Let Ω be a solution of

$$\begin{aligned} \Gamma\,\Omega - \Omega\,\Delta &= \eta\,\theta \\ \bar{\mathrm{d}}\Omega &= (\mathrm{d}\Omega)\,\Delta - (\mathrm{d}\Gamma)\,\Omega + (\mathrm{d}\eta)\,\theta \end{aligned}$$

The equations resulting from acting with d or $\bar{\mathrm{d}}$ on the last equation are satisfied as a consequence of the preceding equations. It follows (Dimakis & M-H 2008) that

$$\phi = \phi_0 - \theta\,\Omega^{-1}\,\eta$$

is a new solution of the ϕ-equation and

$$q = \theta\,\Omega^{-1} \qquad\qquad r = \Omega^{-1}\,\eta$$

satisfy

$$\bar{\mathrm{d}}q = (\mathrm{d}\phi)\,q + \mathrm{d}(q\,\Gamma) \qquad\qquad \bar{\mathrm{d}}r = -r\,(\mathrm{d}\phi) + \mathrm{d}(\Delta\,r)$$

Deformation of the potential

Guided by the pKP example: $\Omega \mapsto \Omega - \omega$. Hence

$$\Gamma\,\Omega - \Omega\,\Delta = \eta\,\theta + c$$
$$\bar{\mathrm{d}}\Omega = (\mathrm{d}\Omega)\,\Delta - (\mathrm{d}\Gamma)\,\Omega + (\mathrm{d}\eta)\,\theta + \gamma$$

where $c := \Gamma\,\omega - \omega\,\Delta$, $\gamma := \bar{\mathrm{d}}\omega - (\mathrm{d}\omega)\,\Delta + (\mathrm{d}\Gamma)\,\omega$.
By straightforward computations, one shows that

$$\phi = \phi_0 - \theta\,\Omega^{-1}\,\eta \qquad q = \theta\,\Omega^{-1} \qquad r = \Omega^{-1}\,\eta$$

constitutes a solution of

$$\mathrm{d}\,\bar{\mathrm{d}}\,\phi + \mathrm{d}\phi\,\mathrm{d}\phi = \mathrm{d}(\,q\,\gamma\,r - q\,\mathrm{d}(c\,r)\,)$$

and

$$\bar{\mathrm{d}}q = (\mathrm{d}\phi)\,q + \mathrm{d}(q\,\Gamma) - q\,\gamma\,\Omega^{-1} - (\mathrm{d}q)\,c\,\Omega^{-1}$$
$$\bar{\mathrm{d}}r = -r\,\mathrm{d}\phi + \mathrm{d}(\Delta\,r) - \Omega^{-1}\,\gamma\,r + \Omega^{-1}\,\mathrm{d}(c\,r)$$

Correspondingly for the g-equation

If ϕ_0 and g_0 satisfy the Miura equation, then ϕ, q, r together with

$$g = (I - \theta\,\Omega^{-1}\,\Gamma^{-1}\eta)\,g_0$$

(I identity matrix) satisfy

$$\bar{\mathrm{d}}g - (\mathrm{d}\phi)\,g = (q\,\gamma + (\mathrm{d}q)\,c)\,\Omega^{-1}\,\Gamma^{-1}\,\Omega\,r\,g_0$$

If $c = 0$, i.e., $\Gamma\,\omega = \omega\,\Delta$, then we have

$$g^{-1} = g_0^{-1}\,(I + \theta\,\Delta^{-1}\,\Omega^{-1}\,\eta)$$

and g satisfies the *extended Miura equation*

$$\bar{\mathrm{d}}g - (\mathrm{d}\phi)\,g = (q\,\gamma\,\Delta^{-1}\,r)\,g$$

which implies the following extension of the g-equation:

$$\mathrm{d}[(\bar{\mathrm{d}}g)\,g^{-1}] = \mathrm{d}(q\,\gamma\,\Delta^{-1}\,r)$$

Via the extended Miura equation we can eliminate ϕ in the previous equations for q and r in favor of g.

Choice of the graded algebra

In the following examples, we specify the graded algebra $\boldsymbol{\Omega}$ to be of the form

$$\boldsymbol{\Omega} = \mathcal{A} \otimes \boldsymbol{\Lambda}\,, \qquad \boldsymbol{\Lambda} = \bigoplus_{i=0}^{2} \boldsymbol{\Lambda}^i$$

where $\boldsymbol{\Lambda}$ is the exterior (Grassmann) algebra of the vector space $\mathbb{C}^2$. It is then sufficient to define d and $\bar{\mathrm{d}}$ on $\mathcal{A}$, since they extend to $\boldsymbol{\Omega}$ in a straightforward way, treating the elements of $\boldsymbol{\Lambda}$ as constants. Moreover, d and $\bar{\mathrm{d}}$ extend to matrices over $\boldsymbol{\Omega}$.
We choose a basis ξ_1, ξ_2 of $\boldsymbol{\Lambda}^1$.

Recovering the pKP example

Let $\mathcal{A}_0$ be the space of smooth complex functions on $\mathbb{R}^3$. We extend it to $\mathcal{A} = \mathcal{A}_0[\partial_x]$. On $\mathcal{A}$ we define

$$\mathrm{d}f = [\partial, f]\xi_1 + \frac{1}{2}[\partial_y + \partial^2, f]\xi_2, \ \bar{\mathrm{d}}f = \frac{1}{2}[\partial_y - \partial^2, f]\xi_1 + \frac{1}{3}[\partial_t - \partial^3, f]\xi_2$$

The maps d and $\bar{\mathrm{d}}$ extend to linear maps on $\boldsymbol{\Omega} = \mathcal{A} \otimes \boldsymbol{\Lambda}$ and moreover to matrices over $\boldsymbol{\Omega}$. We need to choose

$$\Delta = \Gamma = -\partial$$

in order to eliminate explicit operator terms in the linear equations. $c = 0$ iff $\omega_x = 0$. The expression for γ takes the form

$$\gamma = \frac{1}{2}\omega_y \, \xi_1 + \left(\frac{1}{3}\omega_t + \frac{1}{2}\omega_y \, \partial\right) \xi_2$$

We recover all eqs in the pKP section (with $\omega_x = 0$).

Matrix 2d-Toda with self-consistent sources

$\mathcal{A}_0$ complex functions on $\mathbb{R}^2 \times \mathbb{Z}$, smooth in the first two variables.
$\mathcal{A} = \mathcal{A}_0[\mathbb{S}, \mathbb{S}^{-1}]$, $\mathbb{S}$ shift operator in discrete variable $k \in \mathbb{Z}$.

$$\mathrm{d}f = [\mathbb{S}, f]\,\xi_1 + [\partial_y, f]\,\xi_2\,, \qquad \bar{\mathrm{d}}f = [\partial_x, f]\,\xi_1 - [\mathbb{S}^{-1}, f]\,\xi_2$$

For $f \in \mathcal{A}$, we write $f^{\pm}(x, y, k) := f(x, y, k \pm 1)$. We write

$\phi = \varphi\,\mathbb{S}^{-1}$, $q = \tilde{q}\,\mathbb{S}^{-1}$, $r = \mathbb{S}^{-1}\,\tilde{r}$, $\Delta = \Gamma = \mathbb{S}^{-1}$, $\Omega = \tilde{\Omega}\,\mathbb{S}$, $\omega = \tilde{\omega}\,\mathbb{S}$

and set $c = 0$, which is $\tilde{\omega}^{+} = \tilde{\omega}$.

Source-free case: $\mathrm{d}\,\bar{\mathrm{d}}\,\phi + \mathrm{d}\phi\,\mathrm{d}\phi = 0$ becomes

$$\varphi_{xy} - (\varphi^{+} - \varphi)\,(\varphi_y + I) + (\varphi_y + I)\,(\varphi - \varphi^{-}) = 0$$

In the scalar case, in terms of $V := \varphi_y$, this reads
$(\ln(1+V))_x = \varphi^{+} - \varphi + \varphi^{-}$. Differentiating with respect to y, it becomes the *two-dimensional Toda lattice equation*
$(\ln(1+V))_{xy} = V^{+} - 2V + V^{-}$ (Mikhailov '79).
scs-extension and solutions: Hu et al. 2007.

1. $\tilde{\omega}_x = 0$.

$$\varphi_{xy} - (\varphi^+ - \varphi)\,(\varphi_y + I) + (\varphi_y + I)\,(\varphi - \varphi^-) = (\tilde{q}^+\,\tilde{\omega}_y\,\tilde{r}^-)^- - \tilde{q}^+\,\tilde{\omega}_y\,\tilde{r}^-$$

Those of the equations for $\tilde{q}$ and $\tilde{r}$ that do not involve $\tilde{\Omega}$ are

$$\tilde{q}_x = \tilde{q}^+ - \tilde{q} + (\varphi^+ - \varphi)\,\tilde{q}\,, \qquad \tilde{r}_x = \tilde{r} - \tilde{r}^- - \tilde{r}\,(\varphi^+ - \varphi)$$

The Miura-dual is

$$\begin{aligned}
&(g_x\,g^{-1})_y - [g^+\,g^{-1} - (g^+\,g^{-1})^-] = -\tilde{q}^+\,\tilde{\omega}_y\,\tilde{r} + (\tilde{q}^+\,\tilde{\omega}_y\,\tilde{r})^-\,, \\
&\tilde{q}_x = \tilde{q}^+ - \tilde{q} + g_x\,g^{-1}\,\tilde{q}\,, \qquad \tilde{r}_x = \tilde{r} - \tilde{r}^- - \tilde{r}\,g_x\,g^{-1}
\end{aligned}$$

In the scalar case, in terms of $u = \ln g$ this takes the form

$$\begin{aligned}
&u_{xy} - e^{u^+ - u} + e^{u - u^-} = -\tilde{q}^+\,\tilde{\omega}_y\,\tilde{r} + (\tilde{q}^+\,\tilde{\omega}_y\,\tilde{r})^-\,, \\
&\tilde{q}_x = \tilde{q}^+ - \tilde{q} + u_x\,\tilde{q}\,, \qquad \tilde{r}_x = \tilde{r} - \tilde{r}^- - u_x\,\tilde{r}
\end{aligned}$$

2. $\tilde{\omega}_y = 0$.

$$\varphi_{xy} - (\varphi^+ - \varphi)\,(\varphi_y + I) + (\varphi_y + I)\,(\varphi - \varphi^-) = (\tilde{q}\,\tilde{\omega}_x\,\tilde{r}^-)_y$$

Those of the equations for $\tilde{q}$ and $\tilde{r}$ that do not depend on $\tilde{\Omega}$ are

$$\tilde{q}_y = \tilde{q} - \tilde{q}^- - \varphi_y\,\tilde{q}^- \,, \qquad \tilde{r}_y = \tilde{r}^+ - \tilde{r} + \tilde{r}^+\,\varphi_y^+$$

The Miura-dual is

$$\begin{aligned} &(g_x\,g^{-1})_y - [g^+\,g^{-1} - (g^+\,g^{-1})^-] = -(\tilde{q}\,\tilde{\omega}_x\,\tilde{r})_y \,, \\ &\tilde{q}_y = \tilde{q} - g\,(g^{-1}\,\tilde{q})^- \,, \qquad \tilde{r}_y = -\tilde{r} + \tilde{r}^+\,g^+\,g^{-1} \end{aligned}$$

In the scalar case ($m = 1$), in terms of $u = \ln g$, $a = e^{-y}\,g^{-1}\,\tilde{q}\,\tilde{\omega}_x$ and $b = e^y\,g\,\tilde{r}$, this can be expressed as follows,

$$\begin{aligned} &u_{xy} - e^{u^+ - u} + e^{u - u^-} = -(a\,b)_y \,, \\ &a_y + u_y\,a + a^- = 0 \,, \qquad b_y - b\,u_y - b^+ = 0 \,. \end{aligned}$$

Such a system appeared in X. Liu, Y. Zeng, R. Liu 2008.

A matrix version of the discrete KP equation with self-consistent sources

Let $\mathcal{A}_0$ be the space of complex functions of discrete variables $k_0, k_1, k_2 \in \mathbb{Z}$, and $\mathbb{S}_0, \mathbb{S}_1, \mathbb{S}_2$ corresponding shift operators. We extend $\mathcal{A}_0$ to $\mathcal{A} = \mathcal{A}_0[\mathbb{S}_0^{\pm 1}, \mathbb{S}_1^{\pm 1}, \mathbb{S}_2^{\pm 1}]$ and define d and $\bar{\mathrm{d}}$ on $\mathcal{A}$ via

$$\mathrm{d}f = \sum_{i=1}^{2} c_i^{-1} \, [\mathbb{S}_i^{-1}, f] \, \xi_i \, , \qquad \bar{\mathrm{d}}f = \sum_{i=1}^{2} [\mathbb{S}_i^{-1} \mathbb{S}_0, f] \, \xi_i$$

where c_i are constants. Then d and $\bar{\mathrm{d}}$ extend to $\mathbf{\Omega} = \mathcal{A} \otimes \mathbf{\Lambda}$ and to matrices over $\mathbf{\Omega}$. In the following we will use the notation

$$f_{,0} := \mathbb{S}_0 \, f \, \mathbb{S}_0^{-1} \, , \quad f_{,-0} := \mathbb{S}_0^{-1} \, f \, \mathbb{S}_0 \, , \quad f_{,i} := \mathbb{S}_i \, f \, \mathbb{S}_i^{-1} \, , \quad f_{,-i} := \mathbb{S}_i^{-1} \, f \, \mathbb{S}_i$$

We write

$$\Delta = \Gamma = \mathbb{S}_0, \; \Omega = \tilde{\Omega} \, \mathbb{S}_0^{-1}, \; \omega = \tilde{\omega} \, \mathbb{S}_0^{-1}, \; \phi = \varphi \, \mathbb{S}_0, \; q = \tilde{q} \, \mathbb{S}_0, \; r = \mathbb{S}_0 \, \tilde{r}$$

The resulting g-equations are

$$
\begin{aligned}
c_i\left((g_{,i}\, g_{,0}^{-1})_{,j} - g_{,i}\, g_{,0}^{-1}\right) - c_j\left((g_{,j}\, g_{,0}^{-1})_{,i} - g_{,j}\, g_{,0}^{-1}\right) \\
= \quad (c_i-1)\left[(\tilde{q}_{,i}\,(\tilde{\omega}_{,i}-\tilde{\omega})\,\tilde{r}_{,0})_{,j} - \tilde{q}_{,i}\,(\tilde{\omega}_{,i}-\tilde{\omega})\,\tilde{r}_{,0}\right] \\
-(c_j-1)\left[(\tilde{q}_{,j}\,(\tilde{\omega}_{,j}-\tilde{\omega})\,\tilde{r}_{,0})_{,i} - \tilde{q}_{,j}\,(\tilde{\omega}_{,j}-\tilde{\omega})\,\tilde{r}_{,0}\right]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{q}_{,i} &= (c_i-1)^{-1}\left[c_i\, g_{,i}\, g_{,0}^{-1}\,\tilde{q}_{,0} - q\right] - \left[\tilde{q}_{,i}\,(\tilde{\omega}_{,i}-\tilde{\omega})\,\tilde{r}_{,0}\right]\tilde{q}_{,0} \\
&\quad -\tilde{q}_{,i}\,(\tilde{\omega}_{,i}-\tilde{\omega})\,\tilde{\Omega}_{,0,0}^{-1} \\
\tilde{r}_{,-i} &= (c_i-1)^{-1}\left[c_i\,\tilde{r}_{,-0}\, g_{,-0}\, g_{,-i}^{-1} - \tilde{r}\right] - \tilde{r}_{,-0}\left[\tilde{q}_{,-0}\,(\tilde{\omega}-\tilde{\omega}_{,-i})_{,-0}\,\tilde{r}_{,-i}\right] \\
&\quad +\tilde{\Omega}_{,0}^{-1}\,(\tilde{\omega}-\tilde{\omega}_{,-i})\,\tilde{r}_{,-i}
\end{aligned}
$$

Set $\tilde{\omega}_{,1} = \tilde{\omega}$, keep only equations without $\tilde{\Omega}$:

$$
\begin{aligned}
c_2\left((g_{,2}\, g_{,0}^{-1})_{,1} - g_{,2}\, g_{,0}^{-1}\right) - c_1\left((g_{,1}\, g_{,0}^{-1})_{,2} - g_{,1}\, g_{,0}^{-1}\right) \\
= \quad (c_2-1)\left[(\tilde{q}_{,2}\,(\tilde{\omega}_{,2}-\tilde{\omega})\,\tilde{r}_{,0})_{,1} - \tilde{q}_{,2}\,(\tilde{\omega}_{,2}-\tilde{\omega})\,\tilde{r}_{,0}\right],
\end{aligned}
$$

$$
\begin{aligned}
\tilde{q}_{,1} &= (c_1-1)^{-1}\left[c_1\, g_{,1}\, g_{,0}^{-1}\,\tilde{q}_{,0} - \tilde{q}\right], \\
\tilde{r}_{,-1} &= (c_1-1)^{-1}\left[c_1\,\tilde{r}_{,-0}\, g_{,-0}\, g_{,-1}^{-1} - \tilde{r}\right]
\end{aligned}
$$

Scalar discrete KP equation with self-consistent sources

Let $m = 1$, $\tilde{\omega}_{,2} - \tilde{\omega} = K(c_1 - 1)/[c_1(c_2 - 1)]$ constant, and

$$g = \frac{\tau_{,-0}}{\tau} \qquad \tilde{q} = \frac{\rho_{,-0}}{\tau} \qquad \tilde{r} = \frac{\sigma}{\tau_{,-0}}$$

Then

$$(c_1 - 1)\,\tau_{,0}\,\rho_{,1} + \tau_{,0,1}\,\rho - c_1\,\tau_{,1}\,\rho_{,0} = 0$$
$$(c_1 - 1)\,\tau_{,1}\,\sigma_{,0} + \tau\,\sigma_{,0,1} - c_1\,\tau_{,0}\,\sigma_{,1} = 0$$

$$\frac{1}{\tau_{,0}\,\tau_{,1,2}}\Big(c_2\,\tau_{,0,1}\,\tau_{,2} - c_1\,\tau_{,0,2}\,\tau_{,1} - K\,\rho_{,2}\,\sigma_{,0,1}\Big)$$
$$= \left[\frac{1}{\tau_{,0}\,\tau_{,1,2}}\Big(c_2\,\tau_{,0,1}\,\tau_{,2} - c_1\,\tau_{,0,2}\,\tau_{,1} - K\,\rho_{,2}\,\sigma_{,0,1}\Big)\right]_{,-0}$$

The last equation is equivalent to

$$c_2\,\tau_{,0,1}\,\tau_{,2} - c_1\,\tau_{,0,2}\,\tau_{,1} - c_{12}\,\tau_{,0}\,\tau_{,1,2} = K\,\rho_{,2}\,\sigma_{,0,1}$$

with an arbitrary scalar c_{12} that does not depend on the discrete variable k_0.

$\rightarrow$ discrete KP equation with scs (Hu&Wang '06, Doliwa&Lin '14).

The full system of equations for ϕ, q, r, Ω

The following is the abstract system, for which we have a hetero binary Darboux transformation:

$$
\begin{aligned}
&\mathrm{d}\,\bar{\mathrm{d}}\,\phi + \mathrm{d}\phi\,\mathrm{d}\phi = \mathrm{d}(q\,\gamma\,r)\\
&\bar{\mathrm{d}}q = (\mathrm{d}\phi)\,q + \mathrm{d}(q\,\Gamma) + q\,\gamma\,\hat{\Omega}\\
&\bar{\mathrm{d}}r = -r\,\mathrm{d}\phi + \mathrm{d}(\Delta\,r) + \hat{\Omega}\,\gamma\,r\\
&\Delta\,\hat{\Omega} - \hat{\Omega}\,\Gamma = r\,q\\
&\bar{\mathrm{d}}\hat{\Omega} = \mathrm{d}(\hat{\Omega}\Gamma) + (\mathrm{d}r)\,q + \hat{\Omega}\,\gamma\,\hat{\Omega}
\end{aligned}
$$

where

$$
\begin{aligned}
&\bar{\mathrm{d}}\Delta = (\mathrm{d}\Delta)\,\Delta \qquad \bar{\mathrm{d}}\Gamma = \Gamma\,\mathrm{d}\Gamma \qquad \Gamma\,\omega - \omega\,\Delta = 0\\
&\gamma := \bar{\mathrm{d}}\omega - (\mathrm{d}\omega)\,\Delta + (\mathrm{d}\Gamma)\,\omega - \kappa\,\omega - \omega\,\lambda
\end{aligned}
$$

and we have set $\hat{\Omega} := -\Omega^{-1}$.

Let's take a look at this in the pKP hierarchy case.

Extension of the pKP hierarchy

$\mathcal{A}_0$: smooth functions of x and $\mathbf{t} = (t_1, t_2, \dots)$

$$\begin{aligned} \mathrm{d}f &= [\mathcal{E}_{\mu_1}, f]\,\xi_1 + [\mathcal{E}_{\mu_2}, f]\,\xi_2 \\ \bar{\mathrm{d}}f &= [(\mu_1^{-1} - \partial_x)\mathcal{E}_{\mu_1}, f]\,\xi_1 + [(\mu_2^{-1} - \partial_x)\mathcal{E}_{\mu_2}, f]\,\xi_2 \end{aligned}$$

Miwa shift operator: $\mathcal{E}_\mu f = f_{[\mu]}\,\mathcal{E}_\mu$, $f_{[\mu]}(x, \mathbf{t}) = f(x, \mathbf{t} + [\mu])$, $[\mu] = (\mu, \mu^2/2, \mu^3/3, \dots)$. $\Delta = \Gamma = -I_n \partial_x$. Then $\omega_x = 0$ and

$$\begin{aligned} & (\mu_2^{-1} - \phi + \phi_{-[\mu_2]})_{-[\mu_1]}\,(\mu_1^{-1} - \phi + \phi_{-[\mu_1]}) \\ & -(\mu_1^{-1} - \phi + \phi_{-[\mu_1]})_{-[\mu_2]}\,(\mu_2^{-1} - \phi + \phi_{-[\mu_2]}) - (\phi_{-[\mu_1]} - \phi_{-[\mu_2]})_x \\ = \;& \mu_1^{-1} q_{-[\mu_1]}\,(\omega - \omega_{-[\mu_1]})\,r - \mu_1^{-1} q_{-[\mu_1]-[\mu_2]}\,(\omega - \omega_{-[\mu_1]})_{-[\mu_2]}\,r_{-[\mu_2]} \\ & -\mu_2^{-1} q_{-[\mu_2]}\,(\omega - \omega_{-[\mu_2]})\,r + \mu_2^{-1} q_{-[\mu_2]-[\mu_1]}\,(\omega - \omega_{-[\mu_2]})_{-[\mu_1]}\,r_{-[\mu_1]} \end{aligned}$$

$$\begin{aligned} \mu_1^{-1}(q - q_{-[\mu_1]}) - q_x &= (\phi - \phi_{-[\mu_1]})\,q + \mu_1^{-1} q_{-[\mu_1]}\,(\omega - \omega_{-[\mu_1]})\,\hat{\Omega}\,, \\ \mu_1^{-1}(r_{[\mu_1]} - r) - r_x &= -r\,(\phi_{[\mu_1]} - \phi) + \mu_1^{-1}\hat{\Omega}\,(\omega_{[\mu_1]} - \omega)\,r_{[\mu_1]} \end{aligned}$$

$$\hat{\Omega}_x = -r\,q \qquad \mu_1^{-1}(\hat{\Omega}_{[\mu_1]} - \hat{\Omega}) = -r\,q_{[\mu_1]} + \mu_1^{-1}\hat{\Omega}\,(\omega_{[\mu_1]} - \omega)\,\hat{\Omega}$$

The simplest equations

$$q_{t_1} = q_x + q\,\omega_{t_1}\hat{\Omega} \qquad r_{t_1} = r_x + \hat{\Omega}\,\omega_{t_1} r$$
$$q_{t_2} = q_{xx} + 2\,(\phi_{t_1} - q\,\omega_{t_1} r)\,q + q\,\omega_{t_2}\,\hat{\Omega}$$
$$r_{t_2} = -r_{xx} - 2r\,(\phi_{t_1} - q\,\omega_{t_1} r) + \hat{\Omega}\,\omega_{t_2} r$$
$$(\phi_{t_1} - \phi_x - q\,\omega_{t_1} r)_{t_1} = 0$$

Setting $\omega_{t_2} = 0$ and dropping the first two equations, we obtain

$$u_{t_1} - u_x = -2\,(q\,\omega_{t_1} r)_x \qquad q_{t_2} = q_{xx} - u\,q \qquad r_{t_2} = -r_{xx} + r\,u$$

in terms of $u = -2\,(\phi_{t_1} - q\,\omega_{t_1} r)$. Via $t_2 \mapsto -\mathrm{i}\ t_2$, and with the reduction $r = q^\dagger$ (u, ω_{t_1} Hermitian), this becomes

$$u_{t_1} - u_x = -2\,(q\,\omega_{t_1} q^\dagger)_x \qquad \mathrm{i}\ q_{t_2} = q_{xx} - u\,q$$

which is, up to a transformation of the independent variables and absorption of ω, the (2+1)-dimensional *Yajima-Oikawa* system. Now we know it has solutions depending on arbitrary functions of t_1 and we can easily construct such solutions.

Summary

- Integrable systems with self-consistent sources arise via a simple deformation of the potential Ω, which is at the heart of the binary Darboux transformation method, and by dropping some equations that emerge from the linear system.
- The essence of this has been formulated in the framework of *bidifferential calculus*. Choosing any realization of the latter, one should, in general, obtain scs-extensions of the respective integrable equations.
- For the integrable equations addressed so far (KdV, Boussinesq, KP, sine-Gordon, NLS, DS, 2dToda, discrete KP), we recovered all known versions of scs-extensions, even some more, and we generalized them to matrix versions.
- Our approach shows that the appearance of *arbitrary functions* of a single independent variable is a typical feature (on which the “source generation method” of Hu and Wang (2006) is based).

Thanks for your attention !