Integrability of Dispersionless PDEs in 3D and Einstein-Weyl geometry

Boris Kruglikov

University of Tromsø, Norway

joint with E. Ferapontov

Weyl structure

Weyl structure on M^3 is the pair $([g], \mathbb{D})$ consisting of a conformal structure and a linear connection preserving it (we allow any signature, but mostly think of Lorentzian g). Then the condition on $\mathbb{D}$ writes via 1-form ω

$$\mathbb{D}g = \omega \otimes g.$$

A choice of ω is equivalent to a choice of $\mathbb{D}$.

Indeed, denoting by ∇ the Levi-Civita connection, we have $\mathbb{D} = \nabla + \rho(\omega)$, $2\rho(\omega)(X,Y) = \omega(X)Y + \omega(Y)X - g(X,Y)\omega^\sharp$.
In coordinates $\mathbb{D}_i v^j = \nabla_i v^j + \gamma^j_{ik} v^k$, where
$\gamma^j_{ik} = \frac{1}{2}(\omega_k \delta^j_i + \omega_i \delta^j_k - \omega^j g_{ik})$ (raising is done by g).

Under the change $g \mapsto \lambda g$ the form changes so: $\omega \mapsto \omega + d\log\lambda$.
We encode Weyl structures as pairs (g, ω) mod the above gauge.

Einstein-Weyl equation

For the general linear connection $\mathbb{D}$, its Ricci tensor $\mathrm{Ric}_{\mathbb{D}}$ needs not be symmetric. Its skew-symmetric part $\mathrm{Ric}_{\mathbb{D}}^{\mathrm{alt}}$ is proportional to $d\omega$. The symmetric part $\mathrm{Ric}_{\mathbb{D}}^{\mathrm{sym}}$ leads to Einstein-Weyl equation

$$\mathrm{Ric}_{\mathbb{D}}^{\mathrm{sym}} = \Lambda\, g.$$

Here Λ is a function on M. The pair $([g], \mathbb{D})$ is called an Einstein-Weyl structure if the above equation is satisfied.

In particular, for $\omega = 0$ the connection $\mathbb{D}$ is Levi-Civita, and the above is just the Einstein equation. Thus Einstein-Weyl structures are rich generalizations of the Einstein structures. In particular, in 3D all Einstein manifolds are the spaces of constant curvature. But $S^1 \times S^2 = (\mathbb{R}^3 \setminus 0)/\mathbb{Z}$ has a flat Einstein-Weyl structure.

The main goal of this talk is to relate EW structures to integrable systems and to exhibit a way to produce lots of new examples.

Discussion

(1) EW structures are generalizations of Einstein metrics, they are more plentiful, in particular they open up dimension 3 for relativistic applications. Also they generalize locally conformally Einstein manifolds.

(2) The EW is an invariant property of conformal structures, which are more flexible than metric structures.
For instance, it is easier to exhibit EW structures with a symmetry, easier to make deformations.

(3) EW structures in 3D are obtained as reductions of:

- hypercomplex 4D manifolds with triholomorphic vector fields,
- 4D (anti-)selfdual manifolds (ASD) with conformal symmetry.

EW and ODEs

Cartan related EW structures to the geometry of 3rd order ODEs w.r.t. point transformations

$$y''' = F(x, y, y', y'').$$

Denoting $p = y', q = y''$ we have the following (relative) differential invariants ($\mathcal{D} = \partial_x + p\partial_y + q\partial_p + F\partial_q$ is the total derivative):

$$W = \tfrac{1}{6}\mathcal{D}^2 F_q - \tfrac{1}{3}F_q\mathcal{D}F_q - \tfrac{1}{2}\mathcal{D}F_p + \tfrac{2}{27}F_q^3 + \tfrac{1}{3}F_qF_p + F_y$$

$$C = (\tfrac{1}{3}\mathcal{D}F_q - \tfrac{1}{9}F_q^2 - F_p)F_{qq} + \tfrac{2}{3}F_qF_{qp} - 2F_{qy} + F_{pp} + 2W_q$$

(Wünschmann and Cartan invariants).

Provided $W = 0$, $C = 0$ the solution space $\mathcal{S} \simeq \mathbb{R}^3(y, p, q)$ of the ODE carries EW geometry with the conformal structure

$$g = 2\,dy\,dq - \tfrac{2}{3}F_q\,dy\,dp + (\tfrac{1}{3}\mathcal{D}F_q - \tfrac{2}{9}F_q^2 - F_p)\,dy^2 - dp^2$$

and the Weyl potential

$$\omega = \tfrac{2}{3}(F_{qp} - \mathcal{D}F_{qq})\,dy + \tfrac{2}{3}F_{qq}dp.$$

Examples

For the trivial ODE $y''' = 0$ the EW structure is flat: $\omega = 0$, $g = dy^2 + dp^2 + dq^2$. Question: how many 1-forms (connections $\mathbb{D}$) correspond to a particular g, making the pair (g, ω) EW? Answer: at most 4D family; in this case g is conformally flat. For the Euclidean g, the complete solution is $\omega = 2d \log |(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 + A|$ (similar for the Lorenzian signature). As ω is closed, it can be gauge transformed to zero. Then $\mathbb{D}$ is the Levi-Civita connection of a constant curvature metric (Eastwood-Tod).

For the point submaximal symmetric ODE $y''' = \frac{3(y'')^2}{2y'}$ the conformal metric is

$$g = 2dy(dq - \tfrac{q}{p}\, dp) - dp^2.$$

The covector ω is exact, so again the corresponding EW structures are conformal rescalings of the constant curvature metrics.
For general EW structure the map $(g, \omega) \mapsto g$ is one-to-one.

EW and PDEs

It was also known that examples of EW structures can come from solutions of integrable PDEs.

(1) The metric $g = 4dxdt - dy^2 + 4udt^2$ and the covector $\omega = -4u_x dt$ form EW structure provided $u = u(t, x, y)$ satisfies the dKP equation (Dunajski, Mason, Tod)

$$u_{xt} - (uu_x)_x - u_{yy} = 0$$

(2) The metric $g = dx^2 + dy^2 - e^{-u}dt^2$ and the covector $\omega = u_t dt - u_x dx - u_y dy$ form EW structure provided $u = u(t, x, y)$ satisfies the Boyer-Finley equation (Ward, LeBrun)

$$u_{xx} + u_{yy} = (e^u)_{tt}$$

(3) Calderbank found Einstein-Weyl structures from solutions of the gauge field equations with the gauge group $\mathrm{SDiff}(2)$ modelled on Riccati spaces in the class of PDEs related to the generalized Nahm equation.

Dispersionless PDEs

Consider the quasi-linear system of PDEs

$$A(\mathbf{u})\mathbf{u}_x + B(\mathbf{u})\mathbf{u}_y + C(\mathbf{u})\mathbf{u}_t = 0, \qquad (\dagger)$$

where $\mathbf{u} = (u_1, \dots, u_m)^t$ is an m-component vector and A, B, C are $l \times m$ matrices. We assume the system involutive, with the general solution depending on 2 functions of 1 variable.

Systems of type $(\dagger)$ will be referred to as 3D dispersionless PDEs. Typically, they arise as dispersionless limits of integrable soliton equations: The canonical example is the KP equation:

$$u_t - u\,u_x + \epsilon^2 u_{xxx} - w_y = 0, \quad w_x = u_y,$$

which assumes the form $(\dagger)$ in the limit $\epsilon \to 0$.

Notice that $(\dagger)$ is translation invariant, which is the standard requirement for dispersionless PDEs (another approach: scaling limit in independent vars).

Integrability by the method of hydrodynamic reductions

As proposed by Ferapontov and Khusnutdinova, the method of hydrodynamic reductions consists of seeking N-phase solutions

$$\mathbf{u} = \mathbf{u}(R^1, \dots, R^N).$$

The phases (Riemann invariants) $R^i(x, y, t)$ are required to satisfy a pair of commuting equations

$$R^i_y = \mu^i(R) R^i_x, \quad R^i_t = \lambda^i(R) R^i_x,$$

Compatibility of this system writes (commutativity conditions):

$$\frac{\partial_j \mu^i}{\mu^j - \mu^i} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}.$$

Definition

A quasiliner system is called integrable if, for any N, it possesses infinitely many N-component reductions parametrized by N arbitrary functions of 1 variable ($N = 3$ is sufficient).

Example of dKP

Let's rewrite the dKP equation $(u_t - uu_x)_x = u_{yy}$ in the first order (hydrodynamic) form:

$$u_t - uu_x = w_y, \quad u_y = w_x.$$

N-phase solutions are obtained by $u = u(R^1, \dots, R^N)$, $w = w(R^1, \dots, R^N)$, where

$$R^i_y = \mu^i(R) R^i_x, \quad R^i_t = \lambda^i(R) R^i_x.$$

Then

$$\partial_i w = \mu^i \partial_i u, \quad \lambda^i = u + (\mu^i)^2.$$

Functions $u(R)$, $\mu^i(R)$ satisfy the Gibbons-Tsarev equations:

$$\partial_j \mu^i = \frac{\partial_j u}{\mu^j - \mu^i}, \quad \partial_i \partial_j u = \frac{2\partial_i u \partial_j u}{(\mu^j - \mu^i)^2}.$$

This system is involutive and its solutions depend on N functions of 1 variable.

Formal linearization

Given a PDE

$$F(x^i, u, u_{x^i}, u_{x^i x^j}, \dots) = 0,$$

its formal linearization ℓ_F results upon setting $u \to u + \epsilon v$, and keeping terms of the order ϵ. This leads to a linear PDE for v,

$$\ell_F(v) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} F(u + \epsilon v) = 0,$$

In coordinates we have:

$$\ell_F = F_u + F_{u_{x^i}} \mathcal{D}_{x^i} + F_{u_{x^i x^j}} \mathcal{D}_{x^i} \mathcal{D}_{x^j} + \dots .$$

Example: Linearization of the dKP equation, $u_{xt} - (uu_x)_x - u_{yy} = 0$, reads as $v_{xt} - (uv)_{xx} - v_{yy} = 0$. In the latter linear PDE u is the background solution.

This operator arises in a variety of applications:

- *Stability analysis*
- *Symmetries, conservation laws, coverings (= Lax pairs = integrable extensions)*
- *Contact invariants of ODEs, generalized Laplace invariants, Darboux integrable equations*
- *Integrability of ODEs can be seen from the monodromy group of linearized equations*

Main question.

Can one read the integrability (or linearizability) of a given PDE off the geometry of its formal linearization?

Yes, for broad classes of 3D dispersionless second order PDEs.

Types I-IV of PDEs studied:

I. Equations possessing the 'central quadric ansatz':

$$(a(u))_{xx}+(b(u))_{yy}+(c(u))_{tt}+2(p(u))_{xy}+2(q(u))_{xt}+2(r(u))_{yt}=0.$$

Equivalence group: $\mathrm{GL}(3)\times\mathrm{Diff}(\mathbb{R}):\mathbb{R}^3(x,y,t)\times\mathbb{R}^1(u)\circlearrowleft$.

II. Quasilinear wave equations:

$$f_{11}u_{xx}+f_{22}u_{yy}+f_{33}u_{tt}+2f_{12}u_{xy}+2f_{13}u_{xt}+2f_{23}u_{yt}=0,$$

$f_{ij}=f_{ij}(u_x,u_y,u_t)$. Equivalence group: $\mathrm{GL}(4):\mathbb{R}^4(x,y,t,u)\circlearrowleft$.

III. Hirota-type equations:

$$F(u_{xx},u_{xy},u_{yy},u_{xt},u_{yt},u_{tt})=0.$$

Equivalence group: $\mathrm{Sp}(6):T^*\mathbb{R}^3(x,y,t,u_x,u_y,u_t)\circlearrowleft$.

IV. Two-component systems of hydrodynamic type:

$$\mathbf{u}_t=A(\mathbf{u})\mathbf{u}_x+B(\mathbf{u})\mathbf{u}_y,\quad \mathbf{u}=(u_1,u_2)^T.$$

Equiv. group $\mathrm{GL}(3)\times\mathrm{Diff}(\mathbb{R}^2):\mathbb{R}^3(x,y,t)\times\mathbb{R}^2(u_1,u_2)\circlearrowleft$.

Canonical conformal structure

For the equations of the considered type the linearized equation

$$\ell_F(v) = g^{ij} v_{ij} + f^i v_i + c\, v = 0$$

is the second order PDE linear in v. The matrix of higher derivatives represents a symmetric bi-vector $g^{ij} = g^{ij}(u)$ (depending on the 2-jet $j^2 u$ of the solution u) defined up to multiplication by a function.
Thus, provided this matrix is non-degenerate, its inverse $(g_{ij}) = (g^{ij})^{-1}$ determines a canonical conformal metric structure

$$g = g_{ij}\, dx^i\, dx^j,$$

depending on a finite jet of the solution (this encodes the symbol of the equation = dispersion relation). We say that there is a canonical conformal structure on every solution.

A remarkable formula for the Weyl potential

Given a conformal structure $g = g_{ij}(u)dx^i dx^j$ let us introduce the covector $\omega = \omega_s dx^s$ by the universal formula

$$\omega_s = 2g_{sj}\mathcal{D}_{x^k}(g^{jk}) + \mathcal{D}_{x^s}(\ln\det g_{ij}).$$

To interpret this formula, note that the covector ω is given by the identity

$$g^{ij}v_{ij} = \nabla^i\nabla_i v - \tfrac{1}{2}\omega^i\nabla_i v,$$

where ∇ is the Levi-Civita connection. Equivalently, the contracted Christoffel symbols $\Gamma_i = g_{il}g^{jk}\Gamma^l_{jk} = \frac{1}{2}g^{jk}(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk})$ equal to

$$\Gamma_i = -g_{ij}\partial_k g^{jk} - \tfrac{1}{2}\partial_i \log|\det(g_{jk})|,$$

and so (in 3D only!) we relate $\omega_i = -2\Gamma_i$.

Due to dispersionless setup, the formula for ω is not contact invariant, but it is invariant w.r.t. equivalence transformations.

Main results (E. Ferapontov & BK)

Theorem

A second order PDE is linearizable (by a transformation from the natural equivalence group) if and only if the conformal structure g is conformally flat on every solution (has vanishing Cotton tensor).

Theorem

A second order PDE is integrable by the method of hydrodynamic reductions if and only if, on every solution, the conformal structure g satisfies the Einstein-Weyl equations, with the covector $\omega = \omega_s dx^s$ given by the universal formula.

According to a theorem of E. Cartan, the triple $(\mathbb{D}, g, \omega)$ is EW iff there exists a two-parameter family of g-null surfaces that are totally geodesic with respect to $\mathbb{D}$. For our classes of integrable PDEs, these totally geodesic null surfaces are provided by the corresponding dispersionless Lax pair.

Lax pairs

Integrability of the equations of types I-IV above is equivalent to existence of a dispersionless Lax pair

$$S_t = f(S_x, u_x, u_y, u_t), \quad S_y = g(S_x, u_x, u_y, u_t). \qquad (\flat)$$

This means that the compatibility condition $S_{ty} = S_{yt}$ is equivalent to the considered PDE. Lax pairs of this form arise in dispersionless limits of solitonic Lax pairs (Zakharov).

Differentiate $(\flat)$ by x and set $S_x = \lambda$, $u_x = a$, $u_y = b$, $u_t = c$:

$$\lambda_t = f_\lambda \lambda_x + f_a a_x + f_b b_x + f_c c_x,\ \lambda_y = g_\lambda \lambda_x + g_a a_x + g_b b_x + g_c c_x. \quad (\sharp)$$

The vector fields in the extended space $\mathbb{R}^4(x, y, t, \lambda)$

$$X = \frac{\partial}{\partial t} - f_\lambda \frac{\partial}{\partial x} + (f_a a_x + f_b b_x + f_c c_x)\frac{\partial}{\partial \lambda},$$
$$Y = \frac{\partial}{\partial y} - g_\lambda \frac{\partial}{\partial x} + (g_a a_x + g_b b_x + g_c c_x)\frac{\partial}{\partial \lambda},$$

commute iff the compatibility $\lambda_{ty} = \lambda_{yt}$ of $(\sharp)$ holds.

Geometric interpretation à la Twistor theory

Consider the cotangent bundle $Z^6 = T^*\mathbb{R}^3(x, y, t, S_x, S_y, S_t)$ of the soluton $u = u(t, x, y)$. Equations (♭) specify a submanifold $M^4 \subset Z^6$ parametrized by x, y, t, λ. The compatibility of (♭) means this submanifold is coisotropic and for the symplectic form $\omega = dS_x \wedge dx + dS_y \wedge dy + dS_t \wedge dt$ we have:

$$\mathrm{Ker}(\Omega|_{M^4}) = \langle X, Y \rangle.$$

This distribution is tangent to the hypersurface $\lambda = \lambda(x, y, t)$ in M^4.

The two-parameter family of integral leaves of the distribution $\langle X, Y \rangle$ projects to the space $\mathbb{R}^3(x, y, z)$ to a 2-parameter family of null totally geodesic surfaces of the Weyl connection $\mathbb{D}$.

Example of dKP

As an illustration let us consider the dKP equation,

$$u_{xt} - (uu_x)_x - u_{yy} = 0.$$

The corresponding EW structure is as indicated above:
$g = 4dxdt - dy^2 + 4udt^2$, $\omega = -4u_x dt$.

The dispersionless Lax pair is given by vector fields

$$X = \partial_y - \lambda\partial_x + u_x\partial_\lambda, \quad Y = \partial_t - (\lambda^2 + u)\partial_x + (u_x\lambda + u_y)\partial_\lambda,$$

such that the commutativity $[X, Y] = 0$ is equivalent to dKP.

Remark

As a combination of results of Godlinsky-Nurowski, Eastwood-Tod and that of ours, we can conclude that the EW structures coming from PDEs of types I-IV, which are non-linearizable and non-equivalent to dKP have irreducible holonomy.

Another example

The integrable Lagrangian density $u_x u_y u_t$ was obtained in the work of Ferapontov, Khusnutdinova and Tsarev. The corresponding Euler-Lagrange equation

$$u_x u_{yt} + u_y u_{xt} + u_t u_{xy} = 0.$$

is integrable by the method of hydrodynamic reductions.

Conformal structure:
$g = (u_x dx + u_y dy + u_t dt)^2 - 2u_x^2 dx^2 - 2u_y^2 dy^2 - 2u_t^2 dt^2.$
Covector: $\omega = -4\frac{u_x u_{yt}}{u_y u_t}\, dx - 4\frac{u_y u_{tx}}{u_t u_x}\, dy - 4\frac{u_t u_{xy}}{u_x u_y}\, dt.$

The pair (g, ω) determines the connection $\mathbb{D}$ such that the triple $(\mathbb{D}, g, \omega)$ satisfies the Einstein-Weyl condition.

Yet another example

The following equation appeared in the work of Pavlov in the classification of integrable hydrodynamic chains:

$$u_{tt} = \frac{u_{xy}}{u_{xt}} + \frac{1}{6}\eta(u_{xx})u_{xt}^2.$$

Integrability condition = Chazy equation $\eta''' + 2\eta\eta'' = 3(\eta')^2$.

Conformal structure:
$g = 4u_{xt}dxdy - \left(\frac{2}{3}\eta' u_{xt}^4 + s^2\right)dy^2 + 2s\,dydt - dt^2$, here
$s = \frac{1}{3}\eta u_{xt}^2 - \frac{u_{xy}}{u_{xt}}$.
Covector ω equals:

$$\Big[(\tfrac{2}{3}\,u_{tx}\eta + 4u_{xy}u_{tx}^{-2})u_{ttx} + (\tfrac{2}{9}\,u_{tx}^2\eta^2 + \tfrac{8}{3}\,u_{tx}^2\eta' - u_{xy}^2u_{tx}^{-4} - \tfrac{1}{3}\,u_{xy}u_{tx}^{-1}\eta)u_{txx}$$
$$+(\tfrac{1}{9}\,u_{tx}^3\eta\,\eta' + \tfrac{2}{3}\,u_{tx}^3\eta'' - \tfrac{1}{3}\,u_{xy}\eta')u_{xxx} + (u_{xy}u_{tx}^{-3} - \tfrac{1}{3}\,\eta)u_{xxy} - 2\,u_{tx}^{-1}u_{txy}\Big]\,dy$$
$$-\Big[(u_{xy}u_{tx}^{-3} + \tfrac{2}{3}\eta)u_{txx} + \tfrac{1}{3}\eta'\,u_{tx}u_{xxx} - u_{t,x}^{-2}u_{xxy} - 2u_{tx}^{-1}u_{ttx}\Big]\,dt.$$

This structure is EW iff η solves the Chazy equation.

Explicit EW system (M. Dunajski, E. Ferapontov, BK)

According to a theorem of Hitchin, EW equations, considered as a system of PDEs on (g, ω) are integrable.

We construct the explicit Lax pair for EW equations:

$$g = a^2dt^2 - dx^2 + b^2dy^2, \quad \omega = \omega_1 dx + \omega_2 dy + \omega_3 dt$$

is Einstein-Weyl iff

$$\frac{a_{yy}}{ab^2} + \frac{b_{xx}}{b} + \frac{b_{tt}}{a^2b} = \frac{a_xb_x}{ab} + \frac{a_yb_y}{ab^3} + \frac{a_tb_t}{a^3b} + \frac{1}{2a}\left(\frac{\omega_3}{a}\right)_t + \frac{\omega_3^2}{4a^2} + \frac{a}{2}\left(\frac{\omega_1}{a}\right)_x + \frac{\omega_1^2}{4} + \frac{\omega_2a_y}{2ab^2}$$
$$\frac{a_{yy}}{ab^2} + \frac{a_{xx}}{a} + \frac{b_{tt}}{a^2b} = \frac{a_xb_x}{ab} + \frac{a_yb_y}{ab^3} + \frac{a_tb_t}{a^3b} + \frac{1}{2b}\left(\frac{\omega_2}{b}\right)_y + \frac{\omega_2^2}{b^2} + \frac{b}{2}\left(\frac{\omega_1}{b}\right)_x + \frac{\omega_1^2}{4} + \frac{\omega_3b_t}{2a^2b}$$
$$4\frac{a_{xy}}{a} = (\omega_2)_x + (\omega_1)_y + \omega_1\omega_2 + 4\frac{a_yb_x}{ab} - 2\omega_2\frac{b_x}{b}$$
$$4\frac{b_{xt}}{b} = (\omega_3)_x + (\omega_1)_t + \omega_3\omega_1 + 4\frac{a_xb_t}{ab} - 2\omega_3\frac{a_x}{a}$$
$$(\omega_3)_y + (\omega_2)_t + \omega_3\omega_2 - 2\omega_3\frac{a_y}{a} - 2\omega_2\frac{b_t}{b} = 0.$$

Lax pairs for EW (M. Dunajski, E. Ferapontov, BK)

This 5x5 system possesses the Lax pair $[X, Y] = 0$

$$X = \partial_t - a\cos\lambda\ \partial_x + m\ \partial_\lambda, \quad Y = \partial_y - b\sin\lambda\ \partial_x + n\ \partial_\lambda,$$

where λ is the spectral parameter and

$$m = -\tfrac{\omega_2 a}{2b}\sin^2\lambda - \tfrac{\omega_3}{2}\sin\lambda\cos\lambda + \tfrac{1}{2}\sin\lambda(a\omega_1 - 2a_x) + \tfrac{a_y}{b},$$

$$n = \tfrac{\omega_3 b}{2a}\cos^2\lambda + \tfrac{\omega_2}{2}\sin\lambda\cos\lambda - \tfrac{1}{2}\cos\lambda(b\omega_1 - 2b_x) - \tfrac{b_t}{a}.$$

Moreover the system for (g, ω) is hereditary: its symbol carries the quadric (conformal bi-vector), the inverse to which represents g.

Remark: It was an open interest if the Manakov-Santini system represents all the solutions of EW, but apparently it does not.

Integrability in 4D and self-duality

In 4D PDEs of Monge-Ampère type are linearizable iff the corresponding conformal structure is flat on every solution.

Integrable equations of Monge-Ampère type in 4D have the following normal forms (Doubrov-Ferapontov):

- $u_{11} - u_{22} - u_{33} - u_{44} = 0$ (linear wave equation)
- $u_{13} + u_{24} + u_{11}u_{22} - u_{12}^2 = 0$ (second heavenly equation)
- $u_{13} = u_{12}u_{44} - u_{14}u_{24}$ (modified heavenly equation)
- $u_{13}u_{24} - u_{14}u_{23} = 1$ (first heavenly equation)
- $u_{11} + u_{22} + u_{13}u_{24} - u_{14}u_{23} = 0$ (Husain equation)
- $u_{12}u_{34} - \beta u_{13}u_{24} + (\beta - 1)u_{14}u_{23} = 0$ (general heavenly).

Their conformal structures are self-dual on every solution.

Conjecture: A 2nd order dispersionless PDE in 4D is integrable iff the corresponding conformal structure is self-dual on every solution.

According to a theorem of Penrose, the Self-Duality Equations (for metrics) are integrable. Moreover, EW equations are obtained as reductions of the self-duality equations.
Let us give an explicit Lax pair for these PDE.

In Plebanski-like coordinates the conformal metric is

$$g = dwdx + dzdy + pdw^2 + 2qdwdz + rdz^2,$$

where p, q, r are functions of all four variables (x, y, z, w). The self-duality conditions are:

$$p_{xx} + 2q_{xy} + r_{yy} = 0, \quad m_x + n_y = 0,$$
$$n_w - pn_x - qn_y + (p_x + q_y)n = m_z - qm_x - rm_y + (q_x + r_y)m,$$

where

$$m = p_z - q_w + pq_x - qp_x + qq_y - rp_y, \quad n = q_z - r_w + qr_y - rq_y + pr_x - qq_x$$

... continued (and concluded)

The Lax pair $[X, Y] = 0$ with spectral parameter λ is:

$$X = \partial_w - p\partial_x - (q - \lambda)\partial_y + [m - \lambda(p_x + q_y)]\partial_\lambda,$$

$$Y = \partial_z - (q + \lambda)\partial_x - r\partial_y + [n - \lambda(q_x + r_y)]\partial_\lambda.$$

Projecting integral surfaces of the distribution spanned by X and Y from the extended space $\mathbb{R}^5(x, y, z, w, \lambda)$ down to the space of the independent variables $\mathbb{R}^4(x, y, z, w)$, we obtain a 3-parameter family of null surfaces of the corresponding conformal structure g.

This Lax representation of Self-Duality equation also is hereditary: the symbol of the above 3x3 system is equal to Q^3, where $Q = p\,\partial_x^2 + 2q\,\partial_x\partial_y + r\,\partial_y^2 - \partial_x\partial_w - \partial_y\partial_z$ is the conformal bivector. The inverse conformal metric Q^{-1} coincides with g.