Workshop on Geometry of PDEs and Integrability

Integrability in Differential Coverings

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Posing the problem

Let $\mathcal{E} \subset J^\infty(\pi)$ be an equation and $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering. Assume that $\mathcal{E}$ is integrable[1]

$$\boxed{\text{Is } \tilde{\mathcal{E}} \text{ integrable too?}}$$

We shall try to find reasonable answers.

[1] Here and below *integrability* is understood as existence of infinite hierarchies of symmetries and/or conservation laws.

Outline

- ☞ Main definitions
- ☞ The Abelian case
- ☞ The non-Abelian case

References

- ☞ JK & A.M. Vinogradov, *Nonlocal trends in the geometry of differential equations: symmetries, conservation laws, and Bäcklund transformations*, Acta Appl. Math., **15** (1989) 1-2, 161–209.
- ☞ JK, *Some new cohomological invariants of nonlinear differential equations*, Differential Geometry and Its Appl., **2** (1992) no. 4.
- ☞ JK, *Integrability in differential coverings*, `arXiv:1310.1189`

Main definitions (equations)

Consider:

- Locally trivial vector bundle $\pi\colon E \to M$, $\dim M = n$, $\operatorname{rank} \pi = m$;
- Infinitely prolonged equation $\mathcal{E} \subset J^\infty(\pi)$ with the surjection $\pi_\infty\colon \mathcal{E} \to M$.

The bundle π_∞ is endowed with the flat *Cartan connection* $\mathcal{C}\colon D(M) \to D(\mathcal{E})$. The corresponding distribution $\mathcal{C}D(\mathcal{E}) \subset D(\mathcal{E})$ is the *Cartan distribution*.

Equation $\mathcal{E}$ is assumed to be *differentially connected*, i.e., for any linear independent $X_1, \dots, X_n \in D(M)$

$$\mathcal{C}_{X_1}(h) = \dots = \mathcal{C}_{X_n}(h) = 0$$

implies $h = \mathrm{const}$. This means that the 0-horizontal cohomology group $H_h^0(\mathcal{E}) = \mathbb{R}$.

Main definitions (equations)

The projection π_∞ determines the embedding of function algebras $\pi_\infty^*\colon C^\infty(M) \to \mathcal{F}(\mathcal{E})$. Define π_∞-*vertical vector fields*

$$D^v(\mathcal{E}) = \{ X \in D(\mathcal{E}) \mid X|_{C^\infty(M)} = 0 \}$$

and *symmetries* of $\mathcal{E}$

$$\operatorname{sym}\mathcal{E} = \{ S \in D^v(\mathcal{E}) \mid [S, \mathcal{C}D(\mathcal{E})] \subset \mathcal{C}D(\mathcal{E}) \}.$$

Horizontal forms are defined by

$$\Lambda_h^q(\mathcal{E}) = \{ \omega \in \Lambda^q(\mathcal{E}) \mid X \lrcorner \omega = 0, \forall X \in D^v(\mathcal{E}) \}$$

with the *horizontal de Rham differential* $\mathrm{d}_h\colon \Lambda_h^q(\mathcal{E}) \to \Lambda_h^{q+1}(\mathcal{E})$. Elements of $\ker \mathrm{d}_h \subset \Lambda_h^{n-1}(\mathcal{E})$ are *conservation laws*; trivial ones constitute $\operatorname{im} \mathrm{d}_h$. We set

$$\mathrm{CL}(\mathcal{E}) = \Big(\ker \mathrm{d}_h\colon \Lambda_h^{n-1}(\mathcal{E}) \to \Lambda_h^n(\mathcal{E})\Big) / \Big(\operatorname{im} \mathrm{d}_h\colon \Lambda_h^{n-2}(\mathcal{E}) \to \Lambda_h^{n-1}(\mathcal{E})\Big).$$

Coordinates (equations)

- In M: $x^1, \dots, x^n$.
- In the fibers of π: $u^1, \dots, u^m$.
- In $\mathcal{E}$: u^j_σ.

The Cartan connection (*total derivatives*):

$$\mathcal{C}\colon \frac{\partial}{\partial x^i} \mapsto D_{x^i} = \frac{\partial}{\partial x^i} + \sum_{j,\sigma} u^j_{\sigma i} \frac{\partial}{\partial u^j_\sigma}.$$

The Cartan distribution (*Cartan forms*):

$$\omega^j_\sigma = \mathrm{d}u^j_\sigma - \sum_i u^j_{\sigma i}\,\mathrm{d}x^i.$$

Horizontal forms:

$$\omega = \sum a_{i_1,\dots,i_q}\,\mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_q}, \qquad a_{i_1,\dots,i_q} \in \mathcal{F}(\mathcal{E}).$$

Horizontal de Rham differential:

$$\mathrm{d}_h\omega = \sum D_{x^i}(a_{i_1,\dots,i_q})\,\mathrm{d}x^i \wedge \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_q}.$$

Main definitions (coverings)

Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a vector bundle, $\operatorname{rank}\tau = r \le \infty$ (below $r < \infty$ always). It carries a (*differential*) *covering structure* if there is a flat connection $\tilde{\mathcal{C}}$ in the composition $\tilde{\pi}_\infty = \tau \circ \pi_\infty\colon \tilde{\mathcal{E}} \to \mathcal{E}$ such that $\tilde{\mathcal{C}}_X\big|_{\mathcal{F}(\mathcal{E})} = \mathcal{C}_X$ for all $X \in D(M)$ ($\mathcal{F}(\mathcal{E}) \subset \mathcal{F}(\tilde{\mathcal{E}})$ by τ^*).

A covering is *irreducible* if $\tilde{\mathcal{E}}$ is differentially connected.

Two coverings are *equivalent* if

$$\begin{tikzcd} \tilde{\mathcal{E}}_1 \arrow[rr, "f"] \arrow[dr, "\tau_1"'] & & \tilde{\mathcal{E}}_2 \arrow[dl, "\tau_2"] \\ & \mathcal{E} & \end{tikzcd}$$

for a diffeomorphism f such that $f_* \circ \tilde{\mathcal{C}}^1_X = \tilde{\mathcal{C}}^2_X$.

A covering is *trivial* if the system

$$\tilde{\mathcal{C}}_{X_1}(h) = \cdots = \tilde{\mathcal{C}}_{X_n}(h) = 0$$

possesses $\operatorname{rank}\tau$ functionally independent integrals.

Main definitions (coverings)

For two coverings τ_1, τ_2 define their *Whitney product*

$$\begin{tikzcd}
 & \tilde{\mathcal{E}}_1 \times_{\mathcal{E}} \tilde{\mathcal{E}}_2 \arrow[dl, "\tau_1^*(\tau_2)"'] \arrow[dr, "\tau_2^*(\tau_1)"] \arrow[dd, "\tau_1\times\tau_2"] & \\
\tilde{\mathcal{E}}_1 \arrow[dr, "\tau_1"'] & & \tilde{\mathcal{E}}_2 \arrow[dl, "\tau_2"] \\
 & \mathcal{E}. &
\end{tikzcd}$$

by

$$\tilde{\mathcal{C}}_X^{12}(\varphi_1\cdot\varphi_2) = \tilde{\mathcal{C}}_X^{1}(\varphi_1)\cdot\varphi_2 + \varphi_1\cdot\tilde{\mathcal{C}}_X^{2}(\varphi_2), \quad \varphi_i \in \mathcal{F}(\mathcal{E}_i).$$

Proposition

Locally, every covering splits into the Whitney product of a trivial and irredicible ones.

Main definitions (coverings)

Remark

Multidimensional equations ($n > 2$) do not admit nontrivial coverings with $\operatorname{rank}\tau < \infty$. Thus we confine to the case $n = 2$ below.

We say that $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ is an *Abelian covering* if for any fiber-wise linear function $f \in \mathcal{F}(\tilde{\mathcal{E}})$ one has

$$\tilde{\mathcal{C}}_X(f) \in \mathcal{F}(\mathcal{E}) \subset \mathcal{F}(\tilde{\mathcal{E}}).$$

Theorem

Let $\pi_\infty\colon \mathcal{E} \to M$, $\dim M = 2$, be a differentially connected equation. There exists a one-to-one correspondence between equivalence classes of r-dimensional irreducible Abelian coverings τ over $\mathcal{E}$ and r-dimensional vector $\mathbb{R}$-subspaces in $\mathrm{CL}(\mathcal{E})$.

Thus, the Grassmannian $G^r(CL(\mathcal{E}))$ serves as the moduli space for r-dimensional irreducible Abelian coverings.

Coordinates (coverings)

Let $w^1, \dots$ be local coordinates in the fibers of τ. Then the covering structure is given by a system of τ-vertical vector fields

$$X^i = \sum_j X^i_\alpha \frac{\partial}{\partial w^j}, \quad X^i_\alpha \in \mathcal{F}(\tilde{\mathcal{E}}),$$

such that

$$D_{x^i}(X^j) - D_{x^j}(X^i) + [X^i, X^j] = 0, \quad 1 \le i < j \le n. \tag{1}$$

The connection $\tilde{\mathcal{C}}$ is given by

$$\frac{\partial}{\partial x^i} \mapsto \tilde{D}_{x^i} = D_{x^i} + X^i$$

and Equations (1) amount to

$$[\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0$$

for all $i, j = \dots, n$.

Coordinates (coverings)

The pair $(\tilde{\mathcal{E}}, \tilde{\mathcal{C}})$ is isomorphic to the overdetermined system

$$\frac{\partial w^\alpha}{\partial x^i} = X^i_\alpha, \qquad i = 1, \dots, n, \quad \alpha = 1, \dots, r,$$

compatible modulo $\mathcal{E}$. The new unknowns w^α are called *nonlocal variables*.

A covering τ is Abelian if and only if the functions X^i_α are independent of nonlocal variables and locally is of the form

$$\tau = \tau_{\omega_1} \times \dots \times \tau_{\omega_r},$$

where τ_{ω_α} is associated with the conservation law

$$\omega_\alpha = X\,\mathrm{d}x + Y\,\mathrm{d}y.$$

The Abelian case

Let $\tau\colon \mathcal{E} \to M$ be a covering and $\omega \in \mathrm{CL}(\mathcal{E})$. Then $\tau^*\omega \in \mathrm{CL}(\tilde{\mathcal{E}})$. Assume that $\dim M = 2$ and τ is an Abelian irredicible covering. Denote by $\mathcal{L}_\tau \subset \mathrm{CL}(\mathcal{E})$ the subspace associated to τ.

Proposition

Let $\pi_\infty\colon \mathcal{E} \to M$, $\dim M = 2$, be a differentialy connected equation and $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be an irreducible Abelian covering. Let also ω be a conservation law of $\mathcal{E}$. Then $\tau^*\omega$ is trivial if and only if $[\omega] \in \mathcal{L}_\tau$.

As a consequence, we have

Theorem

Let the asumptions of the previous proposition hold. Then if $\mathcal{E}$ possesses infinite number of independent conservation laws then $\tilde{\mathcal{E}}$ has the same property.

The Abelian case

Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering and $S \in \mathrm{sym}\,\mathcal{E}$. We say that S *lifts* to $\tilde{\mathcal{E}}$ if there exists $\tilde{S} \in \mathrm{sym}\,\tilde{\mathcal{E}}$ such that

$$\begin{tikzcd} \mathcal{F}(\mathcal{E}) \arrow[r, hook, "\tau^*"] \arrow[d, "S"'] & \mathcal{F}(\tilde{\mathcal{E}}) \arrow[d, "\tilde{S}"] \\ \mathcal{F}(\mathcal{E}) \arrow[r, hook, "\tau^*"] & \mathcal{F}(\tilde{\mathcal{E}}) \end{tikzcd}$$

Let $\omega \in \mathrm{CL}(\mathcal{E})$. Then $\mathrm{L}_S(\omega)$ is a conservation law as well.

Proposition

Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be an irreducible Abelian covering and $S \in \mathrm{sym}\,\mathcal{E}$. Then S lifts to τ if and only if $\mathrm{L}_S(\mathcal{L}_\tau) \subset \mathcal{L}_\tau$.

The Abelian case

Thus, if S cannot be lifted to τ then its action on $\mathcal{L}_\tau$ generates new conservation laws that do not belong to $\mathcal{L}_\tau$. If the number of such symmetries is infinite the same is the number of conservation laws. The main result in the Abelian case is:

Theorem

Let $\pi_\infty\colon \mathcal{E} \to M$, $\dim M = 2$, *be a differentially connected equation and* $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ *be a finite-dimensional irreducible Abelian covering. Then*:

1. *If* $\mathcal{E}$ *possesses infinite number of conservation laws then the same is valid for* $\tilde{\mathcal{E}}$*.*
2. *If* $\mathcal{E}$ *possesses infinite number of symmetries then* $\tilde{\mathcal{E}}$ *either has the same property, or admits infinite number of conservation laws (or both).*

The non-Abelian case

Let us first clarify what does ‘non-Abelian’ mean? We say that $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ is *strictly non-Abelian* if it is not equivalent to the composition $\tilde{\mathcal{E}} \xrightarrow{\tau_1} \mathcal{E}' \xrightarrow{\tau_2} \mathcal{E}$, where τ_2 is Abelian.

Proposition

If $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ is strictly non-Abelian and ω is a nontrivial conservation law of $\mathcal{E}$ then $\tau^*(\omega)$ is nontrivial as well.

As a consequence, we have

Proposition

If $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ is strictly non-Abelian covering then the map $\tau^*\colon \mathrm{CL}(\mathcal{E}) \to \mathrm{CL}(\tilde{\mathcal{E}})$ is an embedding. In particular, if $\dim \mathrm{CL}(\mathcal{E}) = \infty$ the the same holds for $\mathrm{CL}(\tilde{\mathcal{E}})$.

The non-Abelian case

What hapens to symmetries? First, an old and simple result:

Proposition

Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a finite-dimensional covering and assume that $S \in \operatorname{sym} \mathcal{E}$ possesses a one-parameter group A_t. Then:

- *either S can be lifed to a symmetry $\tilde{S} \in \operatorname{sym} \tilde{\mathcal{E}}$ that is projectible to S by τ,*
- *or the action of the group A_t gives rise to a one-parameter family of coverings $\tau_\lambda\colon \tilde{\mathcal{E}} \to \mathcal{E}$, $\tau_0 = \tau$, with a nonremovable parameter $\lambda \in \mathbb{R}$.*

The situation with symmetries that do not have trajectories is more complicated.

The non-Abelian case

Consider the module $D^v(\Lambda^*(\mathcal{E}))$ of π_∞-vetical $\Lambda^*(\mathcal{E})$-valued derivations and recall that it is a Lie superalgebra with respect to the *Nijenhuis bracket*

$$[\![\cdot,\cdot]\!]\colon D^v(\Lambda^i)\times D^v(\Lambda^j)\to D^v(\Lambda^{i+j}).$$

The equation structure on $\mathcal{E}$ is determined by the *structural element*

$$U_{\mathcal{E}}=\sum_{j,\sigma}(\mathrm{d}u^j_\sigma-\sum_i u^j_{\sigma i}\,\mathrm{d}x^i)\otimes\frac{\partial}{\partial u^j_\sigma}$$

which enjoys $[\![U_{\mathcal{E}},U_{\mathcal{E}}]\!]=0$ and leads to the *$\mathcal{C}$-complex*

$$0\longrightarrow D^v(\mathcal{E})\xrightarrow{\partial_{\mathcal{E}}}D^v(\Lambda^1)\longrightarrow\dots\longrightarrow D^v(\Lambda^i)\xrightarrow{\partial_{\mathcal{E}}}\dots$$

with $\partial_{\mathcal{E}}=[\![U_{\mathcal{E}},\cdot]\!]$. The groups $H^i_{\mathcal{C}}(\mathcal{E})$ are *$\mathcal{C}$-cohomology groups* of $\mathcal{E}$.

The non-Abelian case

Theorem

Let $\mathcal{E}$ be an equation. Then:

1. $H^0_{\mathcal{C}}(\mathcal{E}) = \mathrm{sym}\,\mathcal{E}$.
2. $H^1_{\mathcal{C}}(\mathcal{E})$ *consists of equivalence classes of the equation structure infinitesimal deformations modulo trivial ones.*
3. $H^2_{\mathcal{E}}(\mathcal{C})$ *is the set of obstructions to continuation of infinitesimal deformations to formal ones.*

Given a covering $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$, one has $U_{\tilde{\mathcal{E}}} = U_{\mathcal{E}} + U_\tau$, where

$$U_\tau = \sum_\alpha (\mathrm{d}w^\alpha - \sum_i X_i^\alpha \,\mathrm{d}x^i) \otimes \frac{\partial}{\partial w^\alpha}$$

reflects the covering structure and satisfies the *Maurer-Cartan type equation*

$$\partial_{\mathcal{C}}(U_\tau) + \frac{1}{2}[\![U_\tau, U_\tau]\!] = 0.$$

The non-Abelian case

Cosider the subcomplex of the $\mathcal{C}$-complex for $\tilde{\mathcal{E}}$ consisting of the modules

$$D^g(\Lambda^i(\tilde{\mathcal{E}})) = \{\, Z \in D^v(\Lambda^i(\tilde{\mathcal{E}})) \mid Z|_{\mathcal{F}(\mathcal{E})} = 0 \,\}$$

and denote its cohomology by $H_g^i(\tau)$.

Theorem

Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering. Then:

1. $H_g^0(\tau)$ *consists of gauge symmetries (infinitesimal equivalences) of τ.*
2. $H_g^1(\tau)$ *consists of equivalence classes of the covering structure infinitesimal deformations modulo trivial ones,*
3. $H_g^2(\tau)$ *is the set of obstructions to continuation of infinitesimal deformations to formal ones.*

The non-Abelian case

Let now $S \in \operatorname{sym}\mathcal{E}$ be a symmetry and $\tilde{S} \in D^v(\tilde{\mathcal{E}})$ be its arbitrary lift.

Proposition

The element

$$\bar{U}_\tau = U_\tau + \varepsilon[\![\tilde{S}, (U_{\tilde{\mathcal{E}}})]\!]$$

is an infinitesimal deformation of the covering structure. This deformation is trivial if and only if $\tilde{S} \in \operatorname{sym}\tilde{\mathcal{E}}$*.*

Theorem

Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ *be a covering such that* $H_g^2(\tau) = 0$ *and* $S \in \operatorname{sym}\mathcal{E}$*. Then there exists a formal deformation of the covering structure* τ_ε *in* τ *such that* $\tau_0 = \tau$*.*

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THANK YOU

FOR YOUR ATTENTION