



The spiral minimal surfaces and their Legendre and Weierstrass representations [☆]

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Abstract

A class of spiral minimal surfaces in \mathbb{E}^3 is constructed using a symmetry reduction. The reduction leads to a cubic-nonlinear ODE whose phase portrait is described using an auxiliary Riccati's equation and the Warzewski topological principle for its solutions. The new surfaces are invariant with respect to the composition of rotation and dilation. The solutions are obtained in parametric form through the Legendre and the Weierstrass representations, and also their asymptotic behaviour is described.

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In this paper we construct the two-dimensional minimal surfaces $\Sigma \subset \mathbb{E}^3$ which are invariant with respect to the composition of the dilation centered at the origin and a rotation of space around the origin. We present the solutions in closed form through the Legendre and the Weierstrass representations, and we exhibit their asymptotic behaviour using a nonparametric description. Also, we construct a special solution of the problem.

The paper is organized as follows. In Section 1 we formulate the problem of a symmetry reduction for the minimal surface equation and we express its general solution in two parametric forms. In Section 2 we convert the reduction problem to the auxiliary Riccati equation and classify its solutions. Then in Section 3 we describe the phase portrait of a cubic-nonlinear ODE [1] whose solutions determine the profiles of the minimal surfaces on some cylinder in \mathbb{E}^3 . A special solution of the auxiliary ODE is related to the separatrix on the phase plane. Finally, in Section 4 we construct two classes of the spiral minimal surfaces and indicate their asymptotic approximations. The shape of solutions that belong to the first class resembles the spiral galaxies. The second class of solutions is composed by helicoidal spiral surfaces with exponentially growing helice steps.

[☆] *Subject classification:* Real and complex differential geometry (Riemannian geometry), Classical integrable systems (Algebraic and geometric techniques), Differential equations (Analytic, geometric, and topological methods).

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1. The symmetry reduction problem

Let ξ , η , and z be the Cartesian coordinates in space \mathbb{E}^3 . Consider the surfaces which are locally defined by graphs of functions, $\Sigma = \{z = \chi(\xi, \eta) \mid (\xi, \eta) \in \mathcal{V} \subset \mathbb{R}^2\}$. A two-dimensional surface Σ is minimal (that is, the minimum of area is realized by Σ) if the function χ satisfies the minimal surface equation

$$(1 + \chi_\eta^2)\chi_{\xi\xi} - 2\chi_\xi\chi_\eta\chi_{\xi\eta} + (1 + \chi_\xi^2)\chi_{\eta\eta} = 0. \quad (1)$$

In the sequel, we investigate the properties of Eq. (1) and structures related with it up to its discrete symmetry $\xi \leftrightarrow \eta$ or $z \mapsto -z$.

The full symmetry group of the minimal surface equation is the conformal group; these symmetries incorporate both the contact component and the point transformations generated by the Möbius group as the semi-direct factor. The Lie algebra of point symmetries for Eq. (1) is generated by three translations along the respective coordinate axes, three rotations Ψ_j of the coordinate planes around the origin, and the dilation Δ centered at the origin. It must be noted that the idea to impose a desired isomorphism subgroup in order to reduce the partial differential equation (1) to a second order ODE is standard. Arguments of this kind were already present in Euler's characterization of the catenoid as a unique minimal surface of revolution. In the same way, Catalan proved that the helicoid is a unique nonflat ruled minimal surface. See the review [9] for further discussion.

Let Ψ_1 be the (infinitesimal) rotation of the plane $0\xi\eta$ and consider the generator $\phi = \Psi_1 + \Delta$ of a symmetry of Eq. (1). In [1], the problem of symmetry reduction for Eq. (1) by the composition ϕ of rotation and dilation was posed, although no attempts to find at least one solution were performed. By construction, each solution of this symmetry reduction problem is a minimal surface which is invariant with respect to the one-parametric group of affine motions of \mathbb{E}^3 composed by rotation of the plane $0\xi\eta$ and, simultaneously, dilation of the entire space centered at the origin.

1.1. Solutions via the Legendre transform

The general solution of the reduction problem for Eq. (1) by its symmetry ϕ is obtained [7] in parametric form using the Legendre transformation. Now we describe a two-parametric family of the ϕ -invariant minimal surfaces. Yet, we claim that the reduction provides a special solution that is not incorporated in this family. The present paper is essentially devoted to the description of asymptotic properties of the two classes (regular and special) of these surfaces.

First we recall that the generating section of the symmetry at hand is

$$\phi = \chi - (\xi - \eta)\chi_\xi - (\xi + \eta)\chi_\eta, \quad (2)$$

see [2,8,11] for details. The ϕ -invariance condition for $\chi(\xi, \eta)$ is $\phi = 0$. Next, consider the Legendre transformation

$$\mathcal{L} = \{w = \xi\chi_\xi + \eta\chi_\eta - \chi, \quad p = \chi_\xi, \quad q = \chi_\eta\}. \quad (3)$$

Now we act by the Legendre transformation \mathcal{L} onto the system composed by Eq. (1) and the equation $\phi = 0$. Then from (1) we obtain the linear elliptic equation

$$(1 + p^2)w_{pp} + 2pqw_{pq} + (1 + q^2)w_{qq} = 0, \quad (4a)$$

and from the invariance condition $\phi = 0$ we get the equation

$$w + qw_p - pw_q = 0. \quad (4b)$$

Let (ϱ, ϑ) be the polar coordinates on the plane $0pq$ such that $p = \varrho \cos \vartheta$ and $q = \varrho \sin \vartheta$. Then Eq. (4b) acquires the form $w - \frac{\partial}{\partial \vartheta} w = 0$, whence it follows that $w = \omega(\varrho) \cdot \exp(\vartheta)$. Substituting $w(\varrho, \vartheta)$ in (4a), we obtain the equation

$$\varrho^2 \cdot (1 + \varrho^2)\omega''(\varrho) + \varrho\omega'(\varrho) + \omega(\varrho) = 0. \quad (5)$$

A basis of its solutions is given through

$$\omega_1 = \sqrt{1 + \varrho^2} \cdot \cos \Lambda - \sin \Lambda, \quad \omega_2 = \sqrt{1 + \varrho^2} \cdot \sin \Lambda + \cos \Lambda, \quad (6)$$

where we put $\Lambda = \operatorname{arcoth} \sqrt{1 + \varrho^2} = \frac{1}{2} \ln(\sqrt{1 + \varrho^2} + 1) - \frac{1}{2} \ln(\sqrt{1 + \varrho^2} - 1)$.

The inverse Legendre transformation is $\mathcal{L}^{-1} = \{\xi = w_p, \eta = w_q, \chi = pw_p + qw_q - w\}$. Rewriting \mathcal{L}^{-1} in the polar coordinates ϱ and ϑ , we finally get

$$\begin{aligned} \xi &= \left[\frac{\partial \omega}{\partial \varrho} \cos \vartheta - \frac{\omega}{\varrho} \sin \vartheta \right] \cdot \exp \vartheta, & \eta &= \left[\frac{\partial \omega}{\partial \varrho} \sin \vartheta + \frac{\omega}{\varrho} \cos \vartheta \right] \cdot \exp \vartheta, \\ \chi &= \left[\varrho \cdot \frac{\partial \omega}{\partial \varrho} - \omega \right] \cdot \exp \vartheta. \end{aligned} \tag{7}$$

Formulas (7) provide a parametric representation of the general ϕ -invariant minimal surfaces. Special minimal surfaces are defined by special solutions of Eq. (5) different from the family (6).

Solutions of the symmetry reduction problem are determined by (6)–(7) in closed form but rather implicitly, suggesting no classification for the surfaces. Therefore in what follows we pass to the nonparametric representation of the ϕ -invariant minimal surfaces. Thus we describe their asymptotic properties and point out the special cases.

1.2. The Weierstrass representation of the spiral surfaces

Representation (7) of the minimal surfaces based on the linearizing Legendre transformation (3) is not a unique way to describe open arcwise connected minimal surfaces $\Sigma = \{\vec{\xi} = \xi(p, q)\} \in \mathbb{E}^3$ in parametric form (e.g., we have $p = \chi_\xi$ and $q = \chi_\eta$ in formulas (4)). An alternative is given through the Ennepert–Weierstrass representation (8a), see [4,6,10], where $\vec{\Phi}(\lambda) = \{\Phi_1(\lambda), \Phi_2(\lambda), \Phi_3(\lambda)\}$ is a triple of complex analytic functions that satisfy the constraint $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0$. We see that $\text{Re } \lambda$ and $\text{Im } \lambda$ are the new parameters on the minimal surface Σ .

Theorem 1. *The Ennepert–Weierstrass representation for the general ϕ -invariant minimal surface (7) is*

$$\Sigma_{\Lambda_0} = \left\{ \vec{\xi}(\zeta) \mid \vec{\xi} = \vec{\xi}_0 + \text{Re} \int_0^\zeta \mu \vec{\Phi}(\lambda) d\lambda, \Lambda_0 \in \mathbb{S}^1 \right\}, \tag{8a}$$

where $\zeta \in \mathbb{C}$, $\vec{\xi}_0 \in \mathbb{E}^3$, and $\mu \in \mathbb{R} \setminus \{0\}$ are arbitrary and the Weierstrass data on \mathbb{C} are

$$\vec{\Phi}(\lambda) = \exp(i\Lambda_0) \cdot \exp \lambda \cdot (\sin \lambda, -\cos \lambda, i). \tag{8b}$$

The minimal immersions $\Sigma_{\Lambda_0}: \mathbb{C} \rightarrow \mathbb{E}^3$ are determined up to translation $\vec{\xi}_0$ and homothety μ by formulas (15). The translation $\lambda \mapsto \lambda + \lambda_0$, $\lambda_0 \in \mathbb{R}$, acts on $\vec{\Phi}$ as a dilation of ratio $\exp \lambda_0$ combined with the rotation around the axis Oz by angle λ_0 .

The proof is straightforward using formulas (7) and by setting $\zeta = \vartheta + i\Lambda$. The reasonings are identical for ω_1 and ω_2 , see (6), hence we operate with $\omega_1(\varrho)$ and recover the full generality $\omega(\varrho) = c_1\omega_1 + c_2\omega_2$ in the end.

Proof. Let us substitute the solution $\omega_1(\varrho)$ defined in (6) for ω in (7); we obtain the parametric representation

$$\begin{cases} \xi = \frac{\exp \vartheta}{\varrho} ((\cos \vartheta - \sin \vartheta) \cdot \sqrt{1 + \varrho^2} \cdot \cos \Lambda + (\sin \vartheta + \cos \vartheta) \cdot \sin \Lambda), \\ \eta = \frac{\exp \vartheta}{\varrho} ((\cos \vartheta + \sin \vartheta) \cdot \sqrt{1 + \varrho^2} \cdot \cos \Lambda + (\sin \vartheta - \cos \vartheta) \cdot \sin \Lambda), \\ z = 2 \exp \vartheta \cdot \sin \Lambda. \end{cases} \tag{9}$$

Set $r = 1/\varrho$ and note that $r = \sinh \Lambda$ and $\sqrt{1 + r^2} = \cosh \Lambda$. Then from (9) we obtain

$$\begin{cases} \xi = \exp \vartheta ((\cos \vartheta - \sin \vartheta) \cdot \cosh \Lambda \cdot \cos \Lambda + (\sin \vartheta + \cos \vartheta) \sinh \Lambda \cdot \sin \Lambda), \\ \eta = \exp \vartheta ((\cos \vartheta + \sin \vartheta) \cdot \cosh \Lambda \cdot \cos \Lambda + (\sin \vartheta - \cos \vartheta) \sinh \Lambda \cdot \sin \Lambda), \\ z = 2 \exp \vartheta \cdot \sin \Lambda. \end{cases} \tag{10}$$

Transforming the right-hand sides of (10), we get

$$\begin{cases} \xi = \frac{\sqrt{2}}{2} (\exp(\vartheta - \Lambda) \cdot \cos(\vartheta + \Lambda + \frac{\pi}{4}) + \exp(\vartheta + \Lambda) \cdot \cos(\vartheta - \Lambda + \frac{\pi}{4})), \\ \eta = \frac{\sqrt{2}}{2} (\exp(\vartheta - \Lambda) \cdot \cos(\vartheta + \Lambda - \frac{\pi}{4}) + \exp(\vartheta + \Lambda) \cdot \cos(\vartheta - \Lambda - \frac{\pi}{4})), \\ z = 2 \exp \vartheta \cdot \sin \Lambda. \end{cases} \tag{11}$$

Now we put $\zeta = \vartheta + i\Lambda$ and thence from (11) we obtain the minimal immersion related to ω_1 ,

$$\begin{cases} \xi = \frac{\sqrt{2}}{2} \operatorname{Re}(\exp(\zeta(1+i) + i\frac{\pi}{4}) + \exp(\zeta(1-i) - i\frac{\pi}{4})) = \sqrt{2} \operatorname{Re}(\exp \zeta \cos(\zeta + \frac{\pi}{4})), \\ \eta = \frac{\sqrt{2}}{2} \operatorname{Re}(\exp(\zeta(1+i) - i\frac{\pi}{4}) + \exp(\zeta(1-i) + i\frac{\pi}{4})) = \sqrt{2} \operatorname{Re}(\exp \zeta \cos(\zeta - \frac{\pi}{4})), \\ z = -2 \operatorname{Re}(i \cdot \exp \zeta). \end{cases} \tag{12}$$

Differentiating (12) with respect to ζ and denoting the argument by λ (λ is the integration variable in (8)), we derive the Weierstrass data

$$\begin{aligned} \Phi_1(\gamma) &= \frac{\sqrt{2}}{2} \left((1+i) \cdot \exp\left(\gamma(1+i) + i\frac{\pi}{4}\right) + (1-i) \cdot \exp\left(\gamma(1-i) - i\frac{\pi}{4}\right) \right) = -2 \exp \gamma \sin \gamma, \\ \Phi_2(\gamma) &= \frac{\sqrt{2}}{2} \left((1+i) \cdot \exp\left(\gamma(1+i) - i\frac{\pi}{4}\right) + (1-i) \cdot \exp\left(\gamma(1-i) + i\frac{\pi}{4}\right) \right) = 2 \exp \gamma \cos \gamma, \\ \Phi_3(\gamma) &= -2i \exp \gamma. \end{aligned} \tag{13}$$

Clearly, we have $\Phi_1^2 + \Phi_2^2 + \Phi_3^2 = 0$. Hence we obtain the Weierstrass representation for the minimal surface (10),

$$\vec{\xi} = \vec{\xi}_0 + \operatorname{Re} \left(\operatorname{const} \int_0^\zeta \vec{\Phi}(\lambda) d\lambda \right), \quad \vec{\xi}_0 \in \mathbb{E}^3, \operatorname{const} \in \mathbb{R} \setminus \{0\}, \tag{14}$$

where $\vec{\Phi} = \exp(\lambda) \cdot (\sin \lambda, -\cos \lambda, i)$.

The full generality $\omega = c_1\omega_1 + c_2\omega_2$ in (7) is recovered by setting $\operatorname{const} \in \mathbb{C}$ in (14). Using a homothety, we can always rescale the constant such that it equals $\exp(i\Lambda_0)$ and hence the spiral minimal surfaces Σ_{Λ_0} are parameterized by $\Lambda_0 \in \mathbb{S}^1$. The resulting expressions are as follows. Put $c = \sqrt{c_1^2 + c_2^2}$ such that $\Lambda_0 = \arcsin(c_2/c)$ provided $c \neq 0$. Then the minimal immersion Σ_{Λ_0} is

$$\begin{cases} \xi = c \cdot \exp \vartheta ((\cos \vartheta - \sin \vartheta) \cdot \cosh \Lambda \cdot \cos(\Lambda - \Lambda_0) + (\sin \vartheta + \cos \vartheta) \sinh \Lambda \cdot \sin(\Lambda - \Lambda_0)), \\ \eta = c \cdot \exp \vartheta ((\cos \vartheta + \sin \vartheta) \cdot \cosh \Lambda \cdot \cos(\Lambda - \Lambda_0) + (\sin \vartheta - \cos \vartheta) \sinh \Lambda \cdot \sin(\Lambda - \Lambda_0)), \\ z = 2c \cdot \exp \vartheta \cdot \sin(\Lambda - \Lambda_0). \end{cases} \tag{15}$$

The Weierstrass data (8b) are obtained by differentiating (15) in the form (12) with respect to $\zeta = \vartheta + i\Lambda$; the resulting Weierstrass representation is given through (8a). The proof is complete. \square

The spiral minimal surfaces Σ_0 and $\Sigma_{\pi/2}$ assigned to $\Lambda_0 = 0$ (that is, $\omega = \omega_1$ in (5)) and $\Lambda_0 = \frac{\pi}{2}$ (hence $\omega = \omega_2$), respectively, are given in Figs. 1 and 2. The surface Σ_0 is mirror symmetric with respect to the plane $z = 0$. This surface (up to the central symmetry) intersects the vertical cylinder Q of unit radius through the profile $TS\bar{T}$, see Fig. 3 in Section 4; the analytic and asymptotic properties of this profile are discussed in Example 26 in more detail. The surface $\Sigma_{\pi/2}$ is the harmonic conjugate of Σ_0 ; it contains a vertical line of symmetry.

Remark 2. The analytic vector (8b) can be generalized to a wider class of the Weierstrass data determined by two arbitrary complex analytic functions f and g ,

$$\vec{\Phi}(\lambda) = \exp f(\lambda) \cdot (\sin f(\lambda), -\cos f(\lambda), if(\lambda)) \cdot g(\lambda).$$

The corresponding Weierstrass representation (8a) is the most general under assumption that one of the components (without loss of generality, Φ_3) of the vector $\vec{\Phi}$ does not vanish on an arbitrarily chosen domain in \mathbb{C} .

1.3. Reduction of Eq. (1) to an ODE

Let us return to the system composed by Eq. (1) and the constraint $\phi = 0$. This symmetry reduction leads to one scalar second order ODE with a cubic nonlinearity. Namely, we have

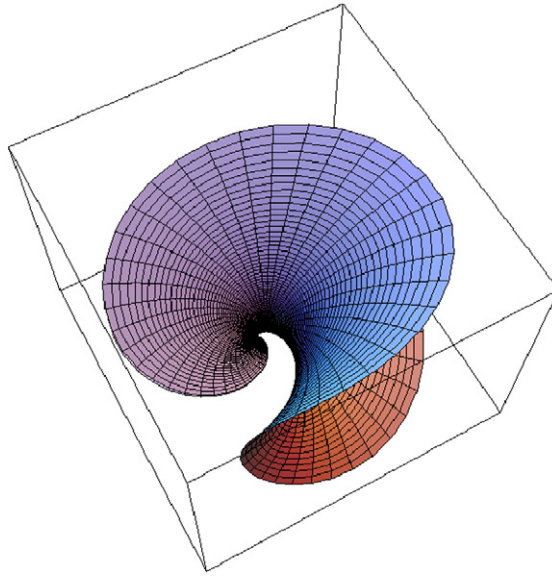


Fig. 1. The spiral surface Σ_0 .

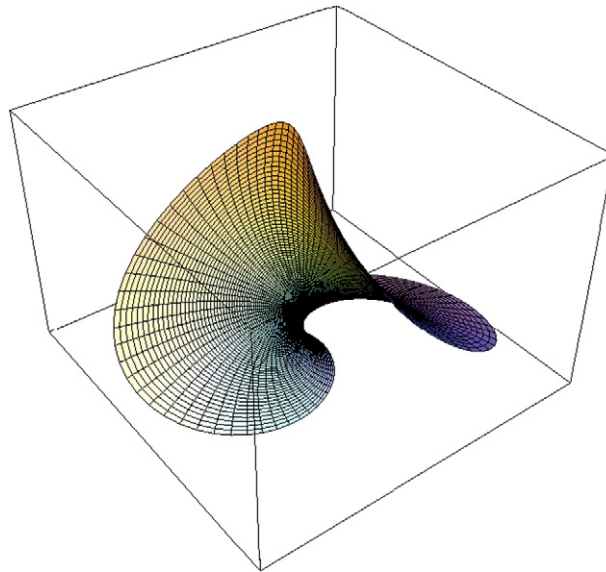


Fig. 2. The minimal surface $\Sigma_{\pi/2}$.

Proposition 3. (See [1].) Let (ρ, θ) be the polar coordinates on the plane $0\xi\eta$ such that (z, ρ, θ) are the cylindric coordinates in space. Consider a minimal surface $\Sigma \subset \mathbb{E}^3$ invariant w.r.t. symmetry (2) of Eq. (1). Then Σ is defined by the formula

$$z = \rho \cdot h(\theta - \ln \rho),$$

where the function $h(q)$ of $q = \theta - \ln \rho$ satisfies the equation

$$h'' \cdot (h^2 + 2) + h - 2h' - (h')^3 - (h - h')^3 = 0. \tag{16}$$

Corollary 4. Assume that a minimal surface Σ is invariant w.r.t. composition ϕ of rotation and dilation, see (2). Let $\Pi = \{z = h(q)|_{\rho=1}\}$ be the profile defined by Σ on the cylinder $Q = \{\rho = 1\} \subset \mathbb{E}^3$, here h is a solution of Eq. (16).

Then the surface Σ is extended from Q along the intersections of the logarithmic spirals $\rho = \text{const} \cdot \exp(\theta)$, $\theta \in \mathbb{R}$, and the cones $z = \rho \cdot h(q)$. The ϕ -invariant surface Σ has a singular point at the origin.

The substitution $h = x$, $h' = y(x)$ maps Eq. (16) to the cubic-nonlinear equation

$$(x^2 + 2)y \frac{dy}{dx} = 2y - x + y^3 - (x - y)^3; \quad (17)$$

Eq. (17) defines the phase portrait of Eq. (16).

Remark 5. Consider the family of equations

$$(x^2 + 2)yy' = -x + \alpha y + y^3 + (y - x)^3 \quad (18)$$

that contains (17) as a particular case when $\alpha = 2$. The change of variables $\frac{1}{u(x)} = (x^2 + 2)y(x)$ in (18) leads to the Abel equation $u' = f_3(x)u^3 + f_2(x)u^2 + f_1(x)u + f_0(x)$, which is integrable in quadratures [12] if $\alpha = 3$. For $\alpha = 2$ we have $f_0 = -\frac{2}{(x^2+2)^2}$, $f_1 = \frac{x}{x^2+2}$, $f_2 = -(3x^2 + 2)$, and $f_3 = x(x^2 + 1)(x^2 + 2)$. The Abel equation can be expressed in the normal form $\eta' = \eta^3 + H(x(\xi))$, where $\eta = \eta(\xi)$, the prime denotes the derivative w.r.t. ξ , and the function $H(x)$ is constructed as follows. Let us make another change of coordinates, $u(x) = \omega(x)\eta(\xi) - \frac{f_2}{3f_3}$, where $\omega(x) = \exp(\int (f_1 - \frac{f_2}{3f_3}) dx) = \frac{(x^2+2)^{1/6}}{x^{2/3}(x^2+1)^{5/6}}$ and $\xi = \xi(x) = \int \frac{(x^2+2)^{4/3}}{x^{1/3}(x^2+1)^{2/3}} dx$. The function $x(\xi)$ is defined by inverting the expression for $\xi(x)$. Then the function $H(x)$ is obtained from the relation

$$f_3 \cdot \omega^3 \cdot H(x) = f_0 + \left(\frac{f_2}{3f_3}\right)'_x - \frac{f_1 f_2}{3f_3} + \frac{2f_2^2}{27f_3^2},$$

whence we finally get

$$H(x) = \left(\frac{x^2 + 1}{x^2 + 2}\right)^{3/2} \cdot \frac{-20}{27x^2(x^2 + 1)^2(x^2 + 2)^2}; \quad (19)$$

note that the inverse function $x(\xi)$ must be substituted for x in (19). Hence we conclude that the coordinate transformation $u(x) = \frac{(x^2+2)^{1/6}}{x^{2/3}(x^2+1)^{5/6}} \cdot \eta(\xi) + \frac{3x^2+2}{3x(x^2+2)(x^2+1)}$ maps the Abel equation to its normal form $\eta' = \eta^3 + H(x(\xi))$.

1.4. Exact solutions of the limit analogue of Eq. (17)

In this subsection we consider the limit analogue of system (17) whose right-hand side contains the terms that provide maximal contribution as $R = \sqrt{x^2 + y^2} \rightarrow \infty$ on $0xy$. To this end, we note that for large R the right-hand side of Eq. (17) is equivalent to its ‘‘cubic’’ component $[y^3 - (x - y)^3] \cdot y^{-1} \cdot x^{-2}$. Let us find exact solutions of the equation

$$\frac{dy}{dx} = \frac{y^3 - (x - y)^3}{yx^2}. \quad (20)$$

Using the substitution $s = \frac{y}{x}$ in (20), we obtain the equation

$$x \frac{ds}{dx} = \frac{2s^3 + 3s - 4s^2 - 1}{s}.$$

Note that $2s^3 - 4s^2 + 3s - 1 = (s - 1)(2s^2 - 2s + 1)$. Therefore we get

$$\int \frac{s ds}{(s - 1)(2s^2 - 2s + 1)} = \frac{1}{2} \ln \frac{(s - 1)^2}{2s^2 - 2s + 1} = \ln |\delta x|, \quad (21)$$

whence we finally obtain the integral $\frac{(s-1)^2}{2s^2-2s+1} = \delta^2 x^2$.

Suppose $\delta = 0$, then we have $s = 1$. Thus we get the solution $y = x$ of approximation (20). In Proposition 21 below we prove that the diagonal is the asymptote for a solution of Eq. (17). If $\delta > 0$, then we solve the quadratic equation

w.r.t. s and obtain

$$s = \frac{1 - \delta^2 x^2 \pm \delta \cdot |x| \cdot \sqrt{1 - \delta^2 x^2}}{1 - 2\delta^2 x^2}.$$

Hence we conclude that

$$\bar{y}_1 = \frac{x \cdot \sqrt{1 - \delta^2 x^2}}{1 - 2\delta^2 x^2} (\sqrt{1 - \delta^2 x^2} + \delta \cdot |x|), \tag{22a}$$

$$\bar{y}_2 = \frac{x \cdot \sqrt{1 - \delta^2 x^2}}{\sqrt{1 - \delta^2 x^2} + \delta \cdot |x|}. \tag{22b}$$

Solutions (22) are defined for $|x| < \frac{1}{\delta}$, $|x| \neq \frac{1}{\delta\sqrt{2}}$. Suppose $x > 0$, then we conclude that

$$\lim_{x \rightarrow \frac{1}{\delta\sqrt{2}} - 0} \bar{y}_1(x) = +\infty.$$

Finally, consider solution (22a) with the Cauchy data (x_0, y_0) , $y_0 \neq x_0$, located on a large circle. Then the above reasonings yield that

$$\delta = \frac{|\frac{y_0}{x_0} - 1|}{\sqrt{2y_0^2 - 2y_0x_0 + x_0^2}}.$$

We claim that the behaviour of solutions of initial equation (17) is correlated with the solutions of limit equation (20), see Remark 25 in Section 3.2.

2. Solutions of the auxiliary Riccati equation

In this section, we start to investigate the phase portrait of Eq. (16). We show that the phase curves are described by solutions of the auxiliary Riccati equation (29a). Using the topological method by Warzewski [5], we conclude that Eq. (29a) has one unstable solution and a large class of stable solutions. On the phase plane for Eq. (16) these solutions are represented by a pair of centrally symmetric separatrices and by trajectories that repulse from the separatrices and have vertical asymptotes at infinity, respectively. This will be discussed in the next section.

Consider the autonomous system associated with (17),

$$\frac{dx}{dt} = 2y + yx^2, \quad \frac{dy}{dt} = 2y - x + y^3 - (x - y)^3. \tag{23}$$

Integral curves for Eq. (17) are assigned to integral trajectories of system (23). Consider the linear change of variables $x = 2(z_1 - z_2)$, $y = 2z_1$. Then system (23) is transformed to

$$\frac{dz_1}{dt} = z_1 + z_2 + 4(z_1^3 + z_2^3), \quad \frac{dz_2}{dt} = -z_1 + z_2 + 4z_2(2z_1^2 - z_1z_2 + z_2^2). \tag{24}$$

The linear approximation for (24) has an unstable focus at the origin; indeed, the eigenvalues are $\lambda_{1,2} = 1 \pm i$. By a Lyapunov’s theorem, nonlinear systems (23) and (24) will exhibit analogous behaviour near the origin.

Next, we use the substitution $z_1 = r \cos \varphi$, $z_2 = r \sin \varphi$ in (24), whence after some transformations we obtain the *triangular* system

$$\dot{r} = r + 4r^3, \tag{25a}$$

$$\dot{\varphi} = -1 + 4r^2 \sin \varphi (\cos \varphi - \sin \varphi). \tag{25b}$$

For any $r_0 > 0$ the Cauchy problem for Eq. (25a) has the solution

$$r(t, r_0) = \frac{r_0 \exp(t)}{\sqrt{1 + 4r_0^2(1 - \exp(2t))}} \tag{26}$$

whenever $r(0, r_0) = r_0$. Assume that $r = r_0 > 0$ at $t = 0$ for a solution $\{r(t), \varphi(t)\}$. From (26) it follows that any solution of system (25) achieves the infinity on the plane $0xy$ at the finite time $t^* = \frac{1}{2} \ln(1 + \frac{1}{4r_0^2})$, and we see that $t^* \rightarrow \infty$ if $r_0 \rightarrow +0$. Obviously, we have $\dot{\varphi} \rightarrow -1$ for $r \rightarrow +0$, that is, the trajectories in a small neighbourhood of the origin are the spirals that unroll clockwise. These reasonings describe the behaviour of the trajectories of systems (23) and (24).

Proposition 6. *The trajectories $z_1(t), z_2(t)$ are transversal to any circle centered at the origin. Therefore the phase curves $\{x(t), y(t)\}$ for Eq. (16) are transversal to any centrally symmetric ellipsis that corresponds to a circle on the plane $0z_1z_2$ (the major axes of these ellipses are located along the diagonal $y = x$). Hence we deduce that system (23) has no cycles and Eq. (16) has no periodic solutions except zero, which corresponds to the stable point on the phase plane. A unique stable point $(0, 0)$ of system (23) is an unstable focus. The spiral-type phase curves for Eq. (16) unroll clockwise around this point. Any trajectory of system (23) that does not coincide with the origin achieves the infinity at a finite time.*

Remark 7. The equations in system (23) do not change under the involution $(x, y) \leftrightarrow (-x, -y)$. Hence for any trajectory of systems (23)–(25) its image under the central symmetry is also a trajectory. Further on, we investigate properties of the phase curves $\{x(t), y(t)\}$ up to this symmetry (or, equivalently, up to the transformation $\varphi \mapsto \varphi + \pi$ of the angular coordinate φ).

2.1. The Riccati equation

Let us divide Eq. (25b) by Eq. (25a). Thus we obtain

$$\frac{d\varphi}{dr} = \frac{-1 + 4r^2 \sin \varphi (\cos \varphi - \sin \varphi)}{r + 4r^3}. \quad (27)$$

Substituting $\tau = 4r^2$ in (27), we get the equation

$$\frac{d\varphi}{d\tau} = \frac{-1 + \tau \sin \varphi (\cos \varphi - \sin \varphi)}{2\tau(1 + \tau)}. \quad (28)$$

After some trigonometric simplifications in the right-hand side of Eq. (28) we arrive at

$$\frac{d\varphi}{d\tau} = \frac{(\sqrt{2} - 1)\tau - 2 - 2\sqrt{2}\tau \sin^2(\varphi - \frac{\pi}{8})}{4\tau(\tau + 1)}. \quad (28')$$

Next, divide both sides in Eq. (28') by $\cos^2(\varphi - \frac{\pi}{8})$ assuming that $\varphi \neq -\frac{3\pi}{8} + \pi n, n \in \mathbb{Z}$. Also, we put $u = \tan(\varphi - \frac{\pi}{8})$ by definition. Hence we finally obtain the Riccati equation

$$\frac{du}{d\tau} = b(\tau) - a(\tau)u^2, \quad (29a)$$

where

$$a(\tau) = \frac{(\sqrt{2} + 1)\tau + 2}{4\tau(\tau + 1)}, \quad b(\tau) = \frac{(\sqrt{2} - 1)\tau - 2}{4\tau(\tau + 1)}. \quad (29b)$$

The Riccati equation (29a) can be further transformed [17] to the Schrödinger equation with the potential b/a and zero energy.

2.2. The Warzewski theorem

In this subsection we study the behaviour of solutions of system (25) as $r \rightarrow +\infty$. First let us recall some definitions [5].

Definition 8. Let $\Omega^0 \subset \Omega$ be an open subset of a domain Ω . Consider the Cauchy problem

$$y' = f(t, y), \tag{30a}$$

$$y(t_0) = y_0. \tag{30b}$$

Suppose $y = y(t)$ is a solution of (30). A point $(t_0, y_0) \in \Omega \cap \partial\Omega^0$ is called an *exit* (resp., *entry*) *point* with respect to Eq. (30a) and the subset Ω^0 if for any solution $y(t)$ satisfying (30b) there is a constant $\varepsilon > 0$ such that $(t, y(t)) \in \Omega^0$ for all t in the interval $t_0 - \varepsilon \leq t < t_0$ (respectively, $t_0 < t \leq t_0 + \varepsilon$). An exit (entry) point (t_0, y_0) for Ω^0 is a *strict exit* (resp., *entry*) *point* if $(t, y(t)) \notin \Omega^0$ whenever $t_0 < t \leq t_0 + \varepsilon$ (respectively, $t_0 - \varepsilon \leq t < t_0$) for some $\varepsilon > 0$. We denote by Ω_e^0 the set of all exit points for the domain Ω^0 , and let Ω_{se}^0 denote the set of all strict exit points.

Now we analyze the behaviour of solutions of Eq. (29a) as $\tau \rightarrow +\infty$. First we note that $\tan \frac{\pi}{8} = \sqrt{2} - 1$; hence we put $u_1 = \sqrt{2} - 1, u_2 = 1 - \sqrt{2}$. The above notation corresponds to the angles $\varphi_1 = \frac{\pi}{4}$ and $\varphi_2 = 0$, respectively. Also, we set $\tau_0 = 2(\sqrt{2} + 1) + \delta_0$, where $\delta_0 > 0$ is arbitrary.

Further, we introduce two closed domains D_1 and D_2 in the right half-plane $\tau > 0$ of the coordinate plane $(\tau; u)$: we let

$$\begin{aligned} D_1 &= \{(\tau; u) \mid \tau \geq \tau_0, 0 \leq u \leq u_1\}, \\ D_2 &= \{(\tau; u) \mid \tau \geq \tau_0, u_2 \leq u \leq 0\}. \end{aligned} \tag{31}$$

Next, we note that the equality $\frac{du}{d\tau} = 0$ is valid on the curves $u(\tau) = \pm\sqrt{b(\tau)/a(\tau)}$. Therefore we define the third domain $D_3 \subset D_1 \cup D_2$ through

$$D_3 = \{(\tau, u) \mid \tau > 2(\sqrt{2} + 1), -\sqrt{b(\tau)/a(\tau)} < u < \sqrt{b(\tau)/a(\tau)}\}.$$

It is easy to check that the inequality $\frac{du}{d\tau} > 0$ holds in the domain D_3 .

By definition, put $f(u, \tau) = b(\tau) - a(\tau)u^2$. Let us describe the inclination field for Eq. (29a) on the lines $u = u_1, u = 0$, and $u = u_2$. We have $f(u_1, \tau) = f(u_2, \tau) = -\frac{1+(\sqrt{2}-1)^2}{2\tau(\tau+1)} < 0$ whenever $\tau \geq \tau_0$, and we also have $f(0, \tau) = b(\tau) > 0$ under the same assumption. This argument shows that all points on the lines $u = 0$ and $u = u_1$ are the strict entry points with respect to $\text{Int } D_1$. Simultaneously, the lines $u = 0$ and $u = u_2$ are composed by the strict exit points w.r.t. the domain $\text{Int } D_2$.

The Warzewski theorem and Example 10 below are borrowed from [5]. Using them, we prove the existence of solutions of Eq. (29a) that do not leave the domains D_1 and D_2 .

Theorem 9. (See [5].) *Let $f(t, y)$ be a continuous function on an open set Ω of points (t, y) , and assume that solutions of system (30a) are uniquely defined by the initial condition (30b). Also, let $\Omega^0 \subset \Omega$ be an open subset such that $\Omega_e^0 = \Omega_{se}^0$. Further let $S \subset \Omega^0$ be a nonempty subset such that*

- *the intersection $S \cap \Omega_e^0$ is a retract of Ω_e^0 , but*
- *the intersection $S \cap \Omega_{se}^0$ is not a retract of S .*

Then there is at least one point $(t_0, y_0) \in S \cap \Omega^0$ such that the graph of solution $y(t)$ of the Cauchy problem (30) is contained in Ω^0 on its maximal right interval of definition.

Example 10. (See [5].) Suppose that y is real and the function $f(t, y)$ in system (30a) is continuous on the set Ω which coincides with the whole plain (t, y) . Let Ω^0 be the strip $|y| < b, -\infty < t < \infty$. The boundary of Ω^0 is contained in Ω and consists of the two lines $y = \pm b$. Assume $f(t, b) > 0$ and $f(t, -b) < 0$ such that $\Omega_e^0 = \Omega_{se}^0 = \partial\Omega^0 \cap \Omega$. Next, let S be the segment $S = \{(t, y) \mid t = 0, |y| \leq b\}$. Then $S \cap \Omega_e^0$ consists of the two points $(0, \pm b)$; the intersection is a retract of the set Ω_e^0 but is not a retract of S . Theorem 9 yields the existence of a point $(0, y_0), |y_0| < b$ such that there is the solution of the Cauchy problem for system (30a) with the initial condition $y(0) = y_0$. This solution satisfies the inequality $|y(t)| < b$ for all $t \geq 0$.

From Example 10 that illustrates Theorem 9 we obtain

Corollary 11. *There is a solution $u = \psi_1(\tau)$ of the Riccati equation (29a) that does not leave the domain $\text{Int } D_1$, which is defined in (31), for all $\tau \geq \tau_0$. Analogously, there is a solution $u = \psi_2(\tau)$ in D_2 that does not leave $\text{Int } D_2$.*

Remark 12. The Warzewski theorem guarantees the existence of a solution ψ_1 in D_1 . We claim that there are infinitely many solutions of this class in our case. The asymptotic expansions of these solutions as $r \rightarrow \infty$ are specified in (39). The convergence of the expansions can be rigorously proved, see Remark 17 in Section 2.3. Also, we claim that the solution ψ_2 in the domain $\text{Int } D_2$ is unique and unstable.

Now we calculate the limits of solutions of the Riccati equation (29a) as $\tau \rightarrow \infty$.

Lemma 13. *Let $u(\tau)$ be a solution of Eq. (29a) which is greater than $u = \psi_1(\tau)$ for all $\tau \geq \tau_0$. Then we have*

$$\lim_{\tau \rightarrow \infty} (u(\tau) - \psi_1(\tau)) = 0. \quad (32)$$

Proof. Assume the converse, $\inf_{\tau \geq \tau_0} (u(\tau) - \psi_1(\tau)) = \Delta_0$, where $\Delta_0 > 0$. Then the difference $w(\tau) = u(\tau) - \psi_1(\tau)$ of two solutions for Eq. (29a) satisfies the equation

$$\frac{dw}{d\tau} = -a(\tau)(u(\tau) + \psi_1(\tau)) \cdot w. \quad (33)$$

Integrating Eq. (33), we obtain

$$w(\tau, \tau_0) = w(\tau_0) \cdot \exp\left(-\int_{\tau_0}^{\tau} a(s)(u(s) + \psi_1(s)) ds\right). \quad (34)$$

Further recall that

$$u(s) + \psi_1(s) \geq u(s) - \psi_1(s) \geq \Delta_0$$

whenever $s \geq \tau_0$ and $a(s) > 0$. Therefore,

$$-\int_{\tau_0}^{\tau} a(s)(u(s) + \psi_1(s)) ds \leq -\Delta_0 \int_{\tau_0}^{\tau} a(s) ds = -\frac{\Delta_0}{4} \left((\sqrt{2} - 1) \ln \frac{\tau + 1}{\tau_0 + 1} + 2 \ln \frac{\tau}{\tau_0} \right).$$

Consequently, we have $\lim_{\tau \rightarrow +\infty} w(\tau, \tau_0) = 0$. This conclusion contradicts the assumption $\Delta_0 > 0$. \square

Similarly we prove that a solution $u(\tau)$ tends to $u = \psi_1(\tau)$ as $\tau \rightarrow +\infty$ if $u(\tau)$ enters the domain D_1 and there it is located under the graph $u = \psi_1(\tau)$.

Lemma 14. *The following equality holds:*

$$\lim_{\tau \rightarrow +\infty} \psi_1(\tau) = \lim_{\tau \rightarrow +\infty} \sqrt{b(\tau)/a(\tau)} = \sqrt{2} - 1 = u_1.$$

Proof. Suppose $\psi_1(\tau) \subset D_1 \cap D_3$ for all $\tau > \tau_0$. Then the solution $\psi_1(\tau)$ grows and is bounded. Therefore there is the limit

$$\lim_{\tau \rightarrow +\infty} \psi_1(\tau) = d_0.$$

Next, let $u = u(\tau)$ be a solution that enters the domain $D_3 \cap D_1$ through the line $u = u_1$ at a large τ_1 and remains in it for all $\tau > \tau_1$. If $d_0 < \sqrt{2} - 1$, then the limit of the solution $u(\tau)$ is $d_1 > d_0$, but the limits d_0 and d_1 must coincide by (32). Consequently, $d_0 = d_1 = \sqrt{2} - 1$. \square

Lemma 15. *The solution $u = \psi_2(\tau)$ remains in the domain $D_2 \setminus D_3$ for all $\tau \geq \tau_0$, and its limit at infinity is*

$$\lim_{\tau \rightarrow +\infty} \psi_2(\tau) = 1 - \sqrt{2}.$$

Proof. If the solution $\psi_2(\tau)$ enters the domain $D_2 \cap D_3$ at some $\tau_1 > \tau_0$ and remains there, then it grows and is bounded. Consequently, there is the limit $\lim_{\tau \rightarrow +\infty} \psi_2(\tau) = d_2$, where $d_2 \in (1 - \sqrt{2}; 0]$. Hence for a large τ_2 and $\tau \geq \tau_2$ we obtain

$$\psi_1(\tau) + \psi_2(\tau) \geq \frac{\sqrt{2} - 1 + d_2}{2}.$$

By definition, put $w(\tau) = \psi_1(\tau) - \psi_2(\tau)$. Then from Eq. (29a) it follows that

$$\frac{dw}{d\tau} = -a(\tau)(\psi_1(\tau) + \psi_2(\tau)) \cdot w,$$

whence we deduce

$$w(\tau) = w(\tau_2) \cdot \exp\left(-\int_{\tau_2}^{\tau} a(s)(\psi_1(s) + \psi_2(s)) ds\right) \leq w(\tau_2) \exp\left(-\frac{\sqrt{2} - 1 + d_2}{2} \cdot \int_{\tau_2}^{\tau} a(s) ds\right). \tag{35}$$

Yet we see that the right-hand side in (35) tends to 0 as $\tau \rightarrow +\infty$. Therefore the left-hand side in (35) must also tend to zero which is impossible. Hence the graph $\psi_2(\tau)$ is contained in $D_2 \setminus D_3$ for large τ , that is, the solution ψ_2 decreases and is bounded from below. This argument shows that

$$\lim_{\tau \rightarrow +\infty} \psi_2(\tau) = \lim_{\tau \rightarrow +\infty} (-\sqrt{b(\tau)/a(\tau)}) = -(\sqrt{2} - 1) = 1 - \sqrt{2}.$$

This completes the proof. \square

Analogously, suppose the graph of a solution enters the domain D_3 and is located above the graph $u = \psi_2(\tau)$. Then it tends to the solution $u = \psi_1(\tau)$. The proof is straightforward. A solution which is less than $u = \psi_2(\tau)$ achieves $-\infty$ at a finite time. Using Eqs. (33) and (34), it can be proved that the solution $u = \psi_2(\tau)$ in the domain $D_2 \setminus D_3$ is unique. The proof is by *reduction ad absurdum*.

2.3. Asymptotic expansions of the solutions $\psi(\tau)$

In this subsection we use the method of undetermined coefficients and obtain the asymptotic expansion in $\frac{1}{\tau}$ for the solution $\psi_1^*(\tau)$ of Eq. (29a) that tends from above to $u_1 = \tan \frac{\pi}{8}$ as $\tau = 4r^2 \rightarrow +\infty$. Also, we get the expansion for $\psi_2(\tau)$ that tends to $u_2 = -\tan \frac{\pi}{8}$ at infinity. We emphasize that the expansion for ψ_1^* provides a solution different from $\psi_1(\tau)$, which exists by Corollary 11. Indeed, the new function ψ_1^* tends to the limit $u_1 = \sqrt{2} - 1$ monotonously decreasing.

Proposition 16. *The solutions $\psi_1^*(\tau)$ and $\psi_2(\tau)$ admit the following asymptotic expansions:*

$$\psi_1^*(\tau) = \sqrt{2} - 1 + \frac{2\sqrt{2}(\sqrt{2} - 1)}{\tau} + O\left(\frac{1}{\tau^2}\right), \tag{36a}$$

$$\psi_2(\tau) = 1 - \sqrt{2} + \frac{2\sqrt{2}(\sqrt{2} - 1)}{3\tau} + O\left(\frac{1}{\tau^2}\right). \tag{36b}$$

The inequalities $\psi_1^*(\tau) > \sqrt{2} - 1$ and $\psi_2(\tau) > 1 - \sqrt{2}$ hold for large τ .

Remark 17. Using the method of majorant series [15,16], we prove that bounded solutions of the Riccati equation (29a) are assigned to expansions (36). These solutions are real analytic in a neighbourhood of the infinity, and the radius of this neighbourhood can be estimated.

2.4. The general case: $u = \psi(r)$.

Now we construct the asymptotic expansions for *all* solutions of (29a) that tend to $\pm(\sqrt{2} - 1)$. This time we use the expansions in r but not in $\tau = 4r^2$.

Let us re-write the Riccati equation (29a) in the form

$$\frac{du}{dr} = \frac{2(\sqrt{2}-1)r^2 - 1}{r(4r^2 + 1)} - \frac{2(\sqrt{2}+1)r^2 + 1}{r(4r^2 + 1)} \cdot u^2. \quad (37)$$

Its right-hand side is real analytic in $\frac{1}{r}$ if $r > \frac{1}{2}$. Suppose $r \geq 1$. Let us use the expansion

$$u(r) = w_0 + \frac{w_1}{r} + \frac{w_2}{r^2} + O\left(\frac{1}{r^3}\right). \quad (38)$$

Substituting (38) for u in (37), we get

$$\begin{aligned} -\frac{w_1}{r^2} - \frac{2w_2}{r^3} - \dots &= \frac{\sqrt{2}-1}{2r} \cdot \left(1 - \frac{1}{4r^2} + \frac{1}{16r^4} - \dots\right) \\ &\quad - \frac{1}{4r^3} \cdot \left(1 - \frac{1}{4r^2} + \frac{1}{16r^4} - \dots\right) - \left(w_0^2 + \frac{2w_0w_1}{r} + \frac{w_1^2 + 2w_0w_2}{r^2} + \dots\right) \\ &\quad \cdot \left(\frac{\sqrt{2}+1}{2r} \cdot \left(1 - \frac{1}{4r^2} + \frac{1}{16r^4} - \dots\right) + \frac{1}{4r^3} \cdot \left(1 - \frac{1}{4r^2} + \frac{1}{16r^4} - \dots\right)\right). \end{aligned}$$

Equating the coefficients of $1/r$ and $1/r^2$, we obtain

$$\sqrt{2}-1 - (\sqrt{2}+1) \cdot w_0^2 = 0, \quad w_1 = (\sqrt{2}+1) \cdot w_0w_1.$$

The former equation has two roots, $w_{0,1} = \sqrt{2}-1$ and $w_{0,2} = 1-\sqrt{2}$. First we let $w_0 = w_{0,1}$; then the root of the second equation is an arbitrary real number! In this case we use the notation $w_1 = C$. Secondly, suppose $w_0 = w_{0,2}$; then a unique root of the second equation is $w_1 = 0$. Now we equate the coefficients of $\frac{1}{r^3}$, whence we get

$$-2w_2 = -\frac{\sqrt{2}-1}{8} - \frac{1}{4} - \frac{w_0^2}{4} - \frac{\sqrt{2}+1}{2} \cdot (w_1 + 2w_0w_2) + \frac{(\sqrt{2}+1)w_0^2}{8}.$$

If $w_0 = \sqrt{2}-1$, then $w_2 = \frac{\sqrt{2}(\sqrt{2}-1)}{2} + \frac{\sqrt{2}+1}{2} \cdot C^2$. Alternatively, if $w_0 = 1-\sqrt{2}$, then $w_2 = \frac{\sqrt{2}(\sqrt{2}-1)}{6}$. This implies that for $C = 0$ we obtain the two asymptotic expansions for the solutions $\psi_1^*(\tau)$ and $\psi_2(\tau)$, which depend on even powers of r and which were previously found in (36). Now we see that all other expansions involve $r = \frac{\sqrt{\tau}}{2}$ explicitly.

By definition, put $D = \frac{1}{2}[\sqrt{2}(\sqrt{2}-1) + (\sqrt{2}+1) \cdot C^2]$. Then for any $u(r)$ such that $\lim_{r \rightarrow +\infty} u(r) = u_1$ we finally have

$$u(r, C) = \sqrt{2}-1 + \frac{C}{r} + \frac{D}{r^2} + O\left(\frac{1}{r^3}\right), \quad C \in \mathbb{R}. \quad (39)$$

An analogue of Remark 17 is also true for (39): using the method of majorant series [15], one readily proves the convergence of expansion (39) for large r to analytic solutions of Eq. (37), and it is also possible to estimate the radius of convergence.

Finally, we formulate the assertion about the behaviour of solutions of the Riccati equation (29a).

Theorem 18. *If $\tau \in (0, 2[\sqrt{2}+1])$, then all solutions of Eq. (29a) decrease monotonously. Suppose $\tau \geq \tau_0$. Then there is the unstable solution $\psi_2(\tau)$ that tends from above to the limit $u_2 = -\tan \frac{\pi}{8}$ as $\tau \rightarrow \infty$; the asymptotic expansion for ψ_2 is given in (36b).*

In the domain D_1 , see (31), there are infinitely many growing solutions $\psi_1(r)$ that tend to $u_1 = \tan \frac{\pi}{8}$ as $r \rightarrow \infty$. These solutions correspond to $C < 0$ in expansion (39).

All solutions which are located between $\psi_2(\tau)$ and $\psi_1(r)$ tend to u_1 as $\tau = 4r^2 \rightarrow \infty$. All solutions which are located under ψ_2 repulse from it; they decrease and achieve $-\infty$ at a finite time.

The solution $\psi_1^*(\tau)$, which is located above the domain D_1 , decreases and tends to u_1 as $\tau \rightarrow \infty$; its expansion is described by (36a) (or by formula (39) with $C = 0$). There are infinitely many other solutions $\psi_1(r)$ which are greater than $\psi_1^*(\tau)$. These solutions also tend to u_1 as $\tau \rightarrow \infty$, and their expansions at infinity are given through (39) with $C > 0$.

3. The phase plane $0xy$

In this section we describe the behaviour of phase curves for Eq. (17). Inverting the transformations introduced on in the beginning of Section 2 and preserving the subscripts of respective functions, we pass from solutions $\psi(\tau)$ of Eq. (29a) to solutions $\{x(t), y(t)\}$ of system (23), and next we obtain solutions $y(x)$ of Eq. (17). For example, the solution $\psi_1^*(\tau)$ is transformed to $y_1^*(x)$ and the separatrix y_2 is assigned to the unstable solution ψ_2 .

The exposition goes along the time t in (23): first we consider a neighbourhood of the origin, next we analyze the behaviour of the trajectories at a finite distance from the origin, and finally we describe their asymptotic expansions at the infinity ($x^2 + y^2 \rightarrow \infty$).

3.1. Extrema and inflection points of phase curves

From Eq. (25a) it follows that system (23) has an unstable focus at the origin. In a small neighbourhood of this point, all trajectories unroll clockwise. By Proposition 6, each trajectory is transversal to the ellipses centered at the origin (we recall that these ellipses correspond to the circles $r = \text{const}$ on the plane $0z_1z_2$); the trajectories achieve the infinity at a finite time.

Proposition 19. All extrema of the phase curves for system (23) are located on the straight line $y = x/2$.

Proof. Solving the equation $P_3(x, y) = (y - x)^3 + y^3 + 2y - x = 0$ with respect to y , we obtain the real root $x/2$. The quotient $P_3(x, y)/(y - x/2) = 2y^2 - 2xy + 2x^2 + 2$ has no real roots for y at any x . \square

Consider the phase trajectories $\{x(t), y(t)\}$ of Eq. (17) in a neighbourhood of the axis $0x$. The parts of the trajectories near the points $(x, 0)$ describe the graphs of solutions $h(q)$ of Eq. (16) near their extrema. The trajectories are approximated with the circles of radius $\varrho(x) = |x| \cdot \frac{x^2+1}{x^2+2}$ centered at the points $(x_0, 0)$, where $x_0 = \frac{|x|}{x^2+2}$. Obviously, we have $x_0 \rightarrow 0$ and $\varrho(x) \rightarrow |x|$ as $|x| \rightarrow \infty$.

Remark 20 (On inflection points). In the first and second quadrants of $0xy$ (and owing to the central symmetry of the phase portrait, in the third and fourth quadrants, respectively) there is the curve \mathcal{I} that consists of the inflection points of the phase curves. The curve \mathcal{I} is composed by two components $\mathcal{I}_{1,2}$ that join in the first quadrant making a cusp; the point of their intersection is the nearest (w.r.t. the Euclidean metric on $0z_1z_2$) inflection point located on the spirals that unroll from the origin.

The first component \mathcal{I}_1 consists of the inflection points where the trajectories start to repulse from the separatrix $y_2(x)$ (the unstable solution y_2 tends from above to the ray $y = x$, see Proposition 21 below) and turn towards their local maxima on the ray $y = x/2$. At the infinity $x^2 + y^2 \rightarrow \infty$, the component \mathcal{I}_1 approaches the diagonal $y = x$ from below.

The second component \mathcal{I}_2 goes left from its intersection with \mathcal{I}_1 and contains the inflection point of the separatrix. Then the curve \mathcal{I}_2 enters the second quadrant. There, it describes the moments when the phase curves, having crossed the $0x$ axis at $x < 0$, repulse from the separatrix and turn upward, possessing the vertical asymptotes at infinity.

3.2. Asymptotes at the infinity

Having achieved the inflection point, the trajectories of system (23) approach the infinity following one of the three schemes below. The expansions for the solutions $\psi_1(\tau)$, $\psi_1^*(r)$, and $\psi_2(\tau)$ of the Riccati equation (29a), which were obtained in the previous section, determine the asymptotes for the phase curves of all the three types.

Proposition 21. *The diagonal $y = x$ is the asymptote of the separatrix $y_2(x)$ in a neighbourhood of infinity; the curve $y = y_2(x)$ approaches the diagonal from above.*

Proof. Using expansion (36b) for the solution $\psi_2(\tau)$ as $\tau = 4r^2 \rightarrow \infty$, and taking into account the transformation from Eq. (23) to Eq. (24), we obtain

$$\begin{aligned} y - x &= 2z_2 = 2r \sin \varphi = 2r \sin \left(\left(\varphi - \frac{\pi}{8} \right) + \frac{\pi}{8} \right) \\ &= 2r \cos \frac{\pi}{8} \cos \left(\varphi - \frac{\pi}{8} \right) \cdot \left(\tan \left(\varphi - \frac{\pi}{8} \right) + \tan \frac{\pi}{8} \right) \\ &= 2r \cos \frac{\pi}{8} (\psi_2(\tau) + \sqrt{2} - 1) / \sqrt{1 + \psi_2^2(\tau)} \\ &= 2r \cos \frac{\pi}{8} \left(\frac{2\sqrt{2}(\sqrt{2} - 1)}{12r^2} + O\left(\frac{1}{r^4}\right) \right) / \sqrt{1 + \psi_2^2(\tau)} \xrightarrow{r \rightarrow \infty} +0. \end{aligned}$$

This argument concludes the proof. \square

The separatrix divides the trajectories in the first (consequently, in the third) quadrant in two classes. The curves y_1^* of the first type repulse from the separatrix and go vertically upward, not leaving the quadrant. Other trajectories dive under the separatrix and achieve the extrema on the ray $y = x/2$. Then they intersect the axis Ox , whence they either turn down and have a vertical asymptote at $-\infty$ or reach the third quadrant. Then the whole situation is reproduced up to the central symmetry.

Now we give a rigorous proof of these properties.

Proposition 22. *The coordinate axis $x = 0$ is the vertical asymptote for the solution y_1^* , which approaches it from the left.*

Proof. Using expansion (36a) and inverting the transformation $Oxy \mapsto Oz_1z_2$, we let $r \rightarrow \infty$ and hence we obtain

$$\begin{aligned} x &= 2(z_1 - z_2) = 2r(\cos \varphi - \sin \varphi) \\ &= -2\sqrt{2}r \sin \left(\varphi - \frac{\pi}{4} \right) = -2\sqrt{2}r \sin \left(\left(\varphi - \frac{\pi}{8} \right) - \frac{\pi}{8} \right) \\ &= -2\sqrt{2} \cos \frac{\pi}{8} \cdot r \left(\tan \left(\varphi - \frac{\pi}{8} \right) - \tan \frac{\pi}{8} \right) \cos \left(\varphi - \frac{\pi}{8} \right) \\ &= -2\sqrt{2} \cos \frac{\pi}{8} \cdot r (\psi_1(\tau) - \sqrt{2} + 1) / \sqrt{1 + \psi_1^2(\tau)} \\ &= -2\sqrt{2} \cos \frac{\pi}{8} \cdot r \left(\frac{2\sqrt{2}(\sqrt{2} - 1)}{4r^2} + O\left(\frac{1}{r^4}\right) \right) / \sqrt{1 + \psi_1^2(\tau)} \rightarrow -0. \end{aligned} \quad (40)$$

According to (25a), the function r grows infinitely along the trajectories. Consequently, the function $R(t) = \sqrt{x^2(t) + y^2(t)}$ also tends to $+\infty$ along the solutions $\{x(t), y(t)\}$. By (40), the coordinate x tends to zero from the left, therefore the axis $x = 0$ is the vertical asymptote of the trajectory $y_1^*(x)$. \square

Let us analyze the asymptotic behaviour of the trajectories $y_1(x)$, which correspond to the solutions $\psi_1(r)$ of the Riccati equation (29a). Using expansion (39) for an arbitrary solution $\psi_1(r)$, we describe the asymptotes on the plane Oxy . We recall that the choice $C > 0$ in formula (39) corresponds to solutions located above the graph $\psi_1^*(\tau)$. The expansion of the function ψ_1^* itself is given at $C = 0$, and the asymptote of its image on the plane Oxy was obtained in Proposition 22. If $C < 0$, then the solutions $\psi_1(r)$ are located between $\psi_1^*(\tau)$ and $\psi_2(\tau)$.

Consider the phase plane Oxy . The solutions of the class y_1 grow infinitely in the second quadrant if $C > 0$. Suppose $C < 0$, then the representatives of this class go to infinity between the axis Oy and the diagonal $y = x$ in the first quadrant; recall that the diagonal is the asymptote of the separatrix y_2 . Using an appropriate modification of the

proof of Proposition 22, we obtain the estimate

$$x = \frac{-2\sqrt{2}r \cos \frac{\pi}{8}}{\sqrt{1+u^2(r,C)}} \left(u(r,C) - \tan \frac{\pi}{8} \right) = \frac{-2\sqrt{2}r \cos \frac{\pi}{8}}{\sqrt{1+u^2(r,C)}} \left(\frac{C}{r} + \frac{D}{r^2} + O\left(\frac{1}{r^3}\right) \right),$$

where $D = \frac{1}{2}[2 - \sqrt{2} + (\sqrt{2} + 1) \cdot C^2]$ and expansion (39) is substituted for $u(r, C)$. Hence we finally obtain

Proposition 23. All solutions of the class y_1 have vertical asymptotes,

$$\lim_{r \rightarrow \infty} x(r, C) = -2\sqrt{2}C \cos^2 \frac{\pi}{8} = -(\sqrt{2} + 1)C.$$

The graphs of solutions contained in the upper half-plane approach these asymptotes from the left.

Remark 24. Without loss of generality, let $C < 0$. Then the vertical straight line $x = -(\sqrt{2} + 1) \cdot C$, which is located in the right half-plane, is the vertical asymptote for two trajectories of system (23). If a solution y_1 is greater than the separatrix y_2 , then it tends to the asymptote from the left. Another trajectory dives under the separatrix and approaches the same asymptote from the right in the fourth quadrant. The constants $C > 0$ describe centrally symmetric curves in the third and second quadrants, respectively.

Remark 25. From Propositions 21–23 it follows that for large R solutions of Eq. (17) are approximated by the exact solutions of limit equation (20). The constants C in (39) and δ in (21) are related by the formula $(\sqrt{2} + 1) \cdot C = \pm(\delta\sqrt{2})^{-1}$.

4. The spiral minimal surfaces

By construction, solutions $h(q)$ of Eq. (16) determine the profiles $\Pi = \{z = h(q) \mid \rho = 1, q \in \mathbb{R}\}$ of the minimal surfaces $\Sigma \subset \mathbb{E}^3$ on the cylinder $Q = \{\rho = 1\}$; we thus have $\Pi = \Sigma \cap Q$. We further recall that a minimal surface in \mathbb{E}^3 is extended from the profile Π in agreement with Corollary 4. Yet, not each surface in \mathbb{E}^3 assigned to a solution of (16) is minimal. Let us study this aspect in more detail.

4.1. The selection rule

In this subsection we formulate a rule that defines which components of the phase trajectories for Eq. (16) provide minimal surfaces. Using this rule, we conclude that the graphs $\{z = \chi(\xi, \eta)\}$ of solutions $\chi = \rho \cdot h(\theta - \ln \rho)$ attach nontrivially to each other.

Let us recall a well-known property of the minimal surfaces [10]. A smooth two-dimensional surface in space is minimal iff at any point its mean curvature H vanishes. Hence each point of the surface is a saddle (of course, we assume that the surface at hand is not a plane).

Further on, we denote by $\frac{\partial}{\partial \theta}$ the respective coordinate vector attached to a point of the cylinder Q . We see that the logarithmic spirals $q = \theta - \ln \rho = \text{const}$ are convex, and at any point of the surface the nonzero curvature vector has a positive projection onto $\frac{\partial}{\partial \theta}$. Therefore we request that the curvature vectors at all points of the profiles $\Pi = \{z = h(q), \rho = 1\} \subset Q$ must have negative projections on $\frac{\partial}{\partial \theta}$.

The above condition upon solutions $h(q)$ of Eq. (16) can be reformulated as follows: $h' \cdot h'' > 0$. This inequality is the rule for selecting the components of the phase curves on the plane $0xy$. Namely, consider a neighbourhood Ω_Γ of a point (x, y) on a phase curve Γ . The minimal surface Σ_Γ is assigned to the component $\Gamma \cap \Omega_\Gamma$ of the curve if the coordinates x and y satisfy the inequality

$$y^3 \cdot (y^3 - (x - y)^3 + 2y - x) > 0.$$

Hence we obtain $y > x/2$ for $y > 0$ and $y < x/2$ whenever $y < 0$, see the proof of Proposition 19. If not all points of a phase curve satisfy this inequality (actually, each curve while it stays near the focus has to pass through the domains where it is not satisfied), then several components of the profiles Π and several minimal surfaces Σ are assigned to this curve. A continuous motion along the phase trajectory Γ corresponds to different (possibly, distant from each other) components of the profile Π .

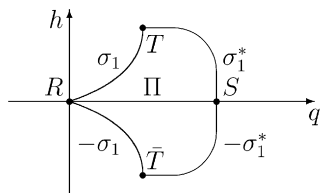


Fig. 3.

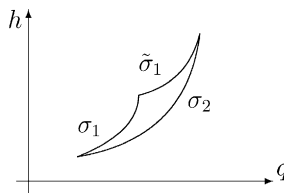


Fig. 4.

In Section 3 we described two classes of trajectories on the plane $0xy$, the curves $y_1(x)$ with vertical asymptotes and the separatrix $y_2(x)$ with the diagonal asymptote $y = x$. Clearly, a small neighbourhood of the focus at the origin corresponds to oscillations of any profile, and some parts of the oscillating curves $z = h(q)$ are prohibited by the selection rule. At infinity, the solutions of first type describe the finite size fragments of the profiles $\Pi_1 = \{z = h(q)\}$; these fragments have the vertical tangent at a finite value of the function h . Conversely, when the separatrix approaches the diagonal asymptote, it defines the exponentially growing profile Π_2 that turns infinitely many times around the cylinder Q .

Example 26. (See [7].) Let us choose the constant $C < 0$ such that $y_1^*(-(\sqrt{2} + 1)C) = 0$. Consider the following two components of the phase curves $\Gamma = \{y = y_1(x, C)\}$:

$$\Gamma_{-\infty} = \{(x, y) \mid -(\sqrt{2} + 1)C \geq x(t) > 0, 0 \geq y = y_1^*(x) \xrightarrow{x \rightarrow +0} -\infty\},$$

$$\Gamma_{+\infty} = \{(x, y) \mid 0 \leq x(t) < -(\sqrt{2} + 1)C, y = y_1(x) \xrightarrow{x \rightarrow -(\sqrt{2}+1)C} +\infty\}.$$

Also, let us use the focus $(0, 0)$, which corresponds to the trivial solution $h \equiv 0$ and hence amounts to the plane $z = 0$ in \mathbb{E}^3 . Using these curves, we construct the closed profiles $\Pi_1(C) \subset Q$, see Fig. 3.

By Corollary 4, the points R and S correspond to the logarithmic spirals on the plane $z = 0$ in \mathbb{E}^3 . Shifting the point R and the components $\pm\sigma_1$ towards S , we make the length $|RS|$ of the profile Π comparable with 2π . Attaching the respective number of the profiles one after another on the cylinder Q , we place the edge R of each profile on the spiral S of the previous profile. Hence the resulting minimal surface becomes self-supporting.

The minimal surface obtained by propagating the subprofile $TST\bar{T}$ along the logarithmic spirals is drawn on Fig. 1.

Example 27. Consider the following parts of the phase curves: a solution $y_1(x)$ that tends to $+\infty$ in the right half-plane, another solution $\tilde{y}_1(x)$ of this class located below the separatrix before the extremum on $y = x/2$, and the separatrix $y_2(x)$ itself. Then we obtain the closed profile with finite cross-section in the upper half-plane $h > 0$. Indeed, we attach the corresponding curves $\sigma_1, \tilde{\sigma}_1$, and σ_2 as shown on Fig. 4.

4.2. The profiles $\Pi = \{z = h_{1,2}(q)\}$ and their approximations

Now we obtain the asymptotic estimates for the profiles Π_1 . In this subsection we assume that the absolute value of the derivative $h' \equiv \frac{dh}{dq}$ (and, possibly, of the function h itself) is large.

Let the constant C in (39) be arbitrary and put

$$D = \frac{1}{2}[2 - \sqrt{2} + C^2 \cdot (\sqrt{2} + 1)]$$

as before. Again, we use expansion (39) as $r \rightarrow \infty$. Then we get the equivalence (\sim)

$$h = x = 2(z_1 - z_2) = 2r(\cos \varphi - \sin \varphi) = \frac{2\sqrt{2} \cos \frac{\pi}{8} \cdot r(\sqrt{2} - 1 - u(r, C))}{\sqrt{1 + u^2(r, C)}} \sim -(\sqrt{2} + 1) \left(C + \frac{D}{r} \right),$$

which holds up to $O(\frac{1}{r^2})$. The derivative h' is expressed through a solution of Eq. (29a) in the following way:

$$h' = y = 2r \cos \varphi = 2r \cos \frac{\pi}{8} \cos \left(\varphi - \frac{\pi}{8} \right) \left(1 - \tan \frac{\pi}{8} \tan \left(\varphi - \frac{\pi}{8} \right) \right) = \frac{2r \cos \frac{\pi}{8} (1 - (\sqrt{2} - 1)u(r, C))}{\sqrt{1 + u^2(r, C)}}.$$

As $r \rightarrow \infty$, the above formula is equivalent (\sim) to

$$\begin{aligned} & 2r \cos^2 \frac{\pi}{8} \left(1 - (\sqrt{2} - 1)^2 - (\sqrt{2} - 1) \left(\frac{C}{r} + \frac{D}{r^2} \right) \right) \\ &= r \left(1 + \frac{\sqrt{2}}{2} \right) (2\sqrt{2} - 2) - \left(1 + \frac{\sqrt{2}}{2} \right) (\sqrt{2} - 1) \left(C + \frac{D}{r} \right) = \sqrt{2}r - \frac{\sqrt{2}}{2} \left(C + \frac{D}{r} \right). \end{aligned}$$

This reasoning shows that for large r we have

$$h \sim -(\sqrt{2} + 1) \left(C + \frac{D}{r} \right), \quad h' \sim \sqrt{2}r - \frac{\sqrt{2}}{2} \left(C + \frac{D}{r} \right).$$

Eliminating r from these formulas, we obtain the differential equation

$$h' = \frac{\sqrt{2}}{2} \frac{h}{\sqrt{2} + 1} - \frac{\sqrt{2}D}{C + \frac{h}{1 + \sqrt{2}}}.$$

By definition, put $g = \frac{h}{\sqrt{2} + 1}$. Then we get

$$(\sqrt{2} + 1)g' = \frac{\sqrt{2}}{2}g - \frac{\sqrt{2}D}{C + g}. \tag{41}$$

Let us consider two cases: $C = 0$ and $C \neq 0$. If $C = 0$, then we see that

$$\frac{(\sqrt{2} + 1)2g \, dg}{\sqrt{2}(g^2 - (2 - \sqrt{2}))} = dq \implies \frac{\sqrt{2} + 1}{\sqrt{2}} \ln |g^2 - (2 - \sqrt{2})| = q + A,$$

where $A = \text{const}$. Resolving this equality w.r.t. g , we obtain

$$g = \pm \sqrt{2 - \sqrt{2} + \exp\left(\frac{\sqrt{2}}{\sqrt{2} + 1}q + A\right)}.$$

Now let $C \neq 0$; then we express $g(q)$ from Eq. (41). Therefore we get $h = (\sqrt{2} + 1)g(q)$ in the form of an integral.

Proposition 28. *Suppose $|h'| \rightarrow \infty$. If $C = 0$, then the asymptotic behaviour of the solution $h_1^*(q)$ of Eq. (16), which is assigned to the phase curve $y_1^*(x)$, is described by the formula*

$$h_1^*(q) \sim \pm \sqrt{2 + \sqrt{2} + (\sqrt{2} + 1)^2 \exp(\sqrt{2}(\sqrt{2} - 1)q + A)},$$

here A is an arbitrary constant. If $C \neq 0$, then the approximation for $h_1^*(q)$ is given implicitly through the integral for Eq. (41),

$$\left| (\sqrt{2} - 1)h + \frac{1}{2}C - B \right|^{2B+C} \cdot \left| (\sqrt{2} - 1)h + \frac{1}{2}C + B \right|^{2B-C} = A \cdot \exp(2(2 - \sqrt{2})Bq),$$

here $A > 0$, $B = \sqrt{\frac{1}{4}C^2 + 2D}$, $C \in \mathbb{R}$, and $D = \frac{1}{2}[2 - \sqrt{2} + C^2 \cdot (\sqrt{2} + 1)]$.

Finally, we use Proposition 21 and describe the profile Π_2 assigned to the separatrix y_2 . The asymptote $y = x$ defines the equivalence $h' \sim h$ as $|h|, |h'| \rightarrow \infty$. Consequently, the profile Π_2 is approximated by solutions of the differential equation $h' = h$.

Proposition 29. *Let h be large, then the function $h_2(q)$ grows exponentially in $q \in \mathbb{R}$: $h_2 \sim A \cdot \exp(q)$, where $A = \text{const}$.*

4.3. Approximations of the minimal surfaces

From Propositions 28 and 29 we obtain the following assertion.

Theorem 30. Let ρ, θ be the polar coordinates on the plane $O\xi\eta$ and put $q = \theta - \ln \rho$, $h = z|_{\rho=1}$.

- (1) For large values of $h'(q)$, there is a ϕ -invariant minimal surface $\Sigma_1^* = \{z = \rho \cdot h_1^*(\theta - \ln \rho)\}$, which is approximated by the graph of the function

$$z = \rho \cdot \sqrt{2 + \sqrt{2} + \frac{(\sqrt{2} + 1)^2 \exp(\sqrt{2}(\sqrt{2} - 1)\theta + A)}{\rho^{\sqrt{2}(\sqrt{2}-1)}}}, \quad A = \text{const.} \quad (42)$$

Also, there is a class of the ϕ -invariant minimal surfaces Σ_1 that correspond to the asymptotic formulas for $h_1(q)$ in Proposition 28. This class contains Σ_1^* as a particular case.

- (2) Suppose that the absolute values $|h_2|$ are sufficiently large such that the phase curve $h_2'(h_2)$ is near to the diagonal $h' = h$. Then there is the ϕ -invariant minimal surface $\Sigma_2 = \{z = \rho \cdot h_2(\theta - \ln \rho)\}$ whose approximation is given by the graph of the function

$$z = A\rho \cdot \exp(\theta - \ln \rho) = A \cdot \exp(\theta), \quad A = \text{const.} \quad (43)$$

The constant $A \in \mathbb{R}$ determines the rotation of the surfaces around the axis Oz . The choice $A < 0$ in (43) defines the reflection symmetry $z \mapsto -z$. Both surfaces $\Sigma_1^*, \Sigma_2 \subset \mathbb{E}^3$ have a singular point at the origin.

Discussion

It must be noted that the spiral minimal surfaces constructed in this paper are neither proper nor complete. Indeed, they are not proper because there are limit point of the surfaces (e.g., the origin which is the center of dilation) that are not contained by the surfaces. Similarly, the minimal immersions $\Sigma: \mathbb{C} \rightarrow \mathbb{E}^3$ are not complete because there are divergent curves in \mathbb{C} such that their image in \mathbb{E}^3 under the Weierstrass transformation has finite length. These two features are not casual because the monotonicity formula [3, Section 2] implies that a complete proper minimal surface in space can not be invariant under a homothety.

We conjecture that the spiral minimal surfaces are relevant in astrophysics for description of celestial objects and their dynamics, cf. the famous ‘Whirlpool’ M51 (NGC 5194) spiral galaxy in *Canes Venatici*. Also, we indicate their application in the fluid dynamics: these surfaces can realize the minimum of the free energy \mathcal{F} of a vortex. Finally, the spiral pattern formation by the minimal surfaces is met in phytocenosis and phyllotaxis [13,14].

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