

THE ABEL SYMPOSIUM 2008

Differential equations: Geometry, Symmetries and Integrability

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Hamiltonian structures for general PDEs

Paul Kersten
Joseph Krasil'shchik
Alexander Verbovetsky
Raffaele Vitolo

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Example: KdV

$$\begin{aligned}u_t = u_{xxx} + 6uu_x &= D_x \delta(u^3 - u_x^2/2) \\ &= (D_{xxx} + 4uD_x + 2u_x)\delta(u^2/2)\end{aligned}$$

$$u_x = v, \quad v_x = w, \quad w_x = u_t - 6uv$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_x = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -6u \\ 0 & 6u & D_t \end{pmatrix} \delta(uw - v^2/2 + 2u^3)$$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix}_x = \begin{pmatrix} 0 & -2u & -D_t - 2v \\ 2u & D_t & -12u^2 - 2w \\ -D_t + 2v & 12u^2 + 2w & 8uD_t + 4u_t \end{pmatrix} \delta(-3u^2/2 - w/2)$$

S. P. Tsarev, *The Hamilton property of stationary and inverse equations of condensed matter mechanics and mathematical physics*, Math. Notes **46** (1989), 569–573

Example: Camassa-Holm equation

$$u_t - u_{txx} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0$$

$$m_t + um_x + 2u_x m = 0, \quad m - u + u_{xx} = 0$$

$$m_t = -um_x - 2u_x m = B_1 \delta(\mathcal{H}_1) = B_2 \delta(\mathcal{H}_2)$$

where

$$B_1 = -(mD_x + D_x m), \quad \mathcal{H}_1 = \frac{1}{2} \int mu \, dx$$

$$B_2 = D_x^3 - D_x, \quad \mathcal{H}_2 = \frac{1}{2} \int (u^3 + uu_x^2) \, dx.$$

\mathcal{H}_1 and \mathcal{H}_2 are viewed as functionals of m and not of u ,
with $u = (1 - D_x^2)^{-1}m$.

Example: Kupershmidt deformation

B. Kupershmidt, *KdV6: An integrable system*, Phys. Lett. A **372** (2008), 2634–2639

$$u_t = f(t, x, u, u_x, u_{xx}, \dots)$$

A_1, A_2 are compatible Hamiltonian operators

H_1, H_2, \dots is a Magri hierarchy of conserved densities

$$D_t(H_i) = 0, \quad A_1 \delta(H_i) = A_2 \delta(H_{i+1}).$$

$$u_t = f - A_1(w), \quad A_2(w) = 0 \tag{1}$$

The KdV6 equation

(A. Karasu-Kalkanli, A. Karasu, A. Sakovich, S. Sakovich, and R. Turhan, *A new integrable generalization of the Korteweg-de Vries equation*, arXiv:0708.3247)

$$u_t = u_{xxx} + 6uu_x - w_x, \quad w_{xxx} + 4uw_x + 2u_x w = 0$$

Theorem (Kupershmidt)

H_1, H_2, \dots are conserved densities for (1).

Notation: infinite jet space

The jet space J^∞ with coordinates x_i, u_σ^j

$D_i = \partial_{x_i} + \sum_{j,\sigma} u_{\sigma i}^j \partial_{u_\sigma^j}$ are total derivatives

D_i span the Cartan distribution

$E_\varphi = \sum_j \varphi^j \partial_{u^j} + \sum_{ji} D_i(\varphi^j) \partial_{u_i^j} + \dots$ is an evolutionary field,
 $\varphi = (\varphi^1, \dots, \varphi^m)$ is a vector function on J^∞

$\ell_f = \left\| \sum_\sigma \partial_{u_\sigma^j}(f_i) D_\sigma \right\|$ is the linearization

of a vector function f on J^∞ , $\ell_f(\varphi) = E_\varphi(f)$

$\Delta^* = \left\| \sum_\sigma (-1)^\sigma D_\sigma a_\sigma^{ji} \right\|$, if $\Delta = \left\| \sum_\sigma a_\sigma^{ij} D_\sigma \right\|$,

the adjoint \mathcal{C} -differential operator

Notation: differential equations

Let $F_k(x_i, u_\sigma^j) = 0$, $k = 1, \dots, l$, be a system of equations

Relations $F = 0$, $D_\sigma(F) = 0$ define its infinite prolongation $\mathcal{E} \subset J^\infty$

$\ell_{\mathcal{E}} = \ell_F|_{\mathcal{E}}$ is the linearization of the equation \mathcal{E}

E_φ is a symmetry of \mathcal{E} if $E_\varphi(F)|_{\mathcal{E}} = \ell_{\mathcal{E}}(\varphi) = 0$, $\text{Sym}(\mathcal{E}) = \ker \ell_{\mathcal{E}}$
 φ is its generating function

Vector function $S = (S^1, \dots, S^n)$ on \mathcal{E} is a conserved current if $\sum_i D_i(S^i) = 0$ on \mathcal{E}

A conserved current is trivial if

$$S^i = \sum_{j < i} D_j(T^{ji}) - \sum_{i < j} D_j(T^{ij})$$

Conservation laws of \mathcal{E} are the conserved currents modulo trivial ones.

Generating function of a conservation law:

$$\psi = (\psi_1, \dots, \psi_m) = \Delta^*(1), \text{ where } \sum_i D_i(S^i) = \Delta(F) \text{ on } J^\infty$$

$$\ell_{\mathcal{E}}^*(\psi) = 0, \quad \text{CL}(\mathcal{E}) \subset \ker \ell_{\mathcal{E}}^*$$

ℓ^* -covering

Cotangent bundle to an equation

the equation $\mathcal{L}^*(\mathcal{E})$ is given by the system

$$\begin{array}{ccc} \ell^*\text{-covering: } \mathcal{L}^*(\mathcal{E}) & \ell_F^*(p) = 0, & F = 0 \\ \downarrow \tau^* & & \\ \mathcal{E} & & \end{array}$$

τ^* is the natural projection $\tau^*: (u_\sigma^j, p_\sigma^k) \mapsto (u_\sigma^j)$
variables p_σ^k along the fibers of the covering are odd.

$\langle F, p \rangle$ is the Lagrangian for $\mathcal{L}^*(\mathcal{E})$

Theorem

There is a natural 1-1 correspondence between the symmetries of \mathcal{E} and the conservation laws of $\mathcal{L}^(\mathcal{E})$ linear along the fibers of τ^* .*

φ is a symmetry $\Rightarrow \ell_F(\varphi) = \Delta(F)$, φ_Δ corresponds to Δ^*
 $(\varphi, \varphi_\Delta)$ is the conservation law

Dictionary

Manifold M

PDE \mathcal{E}

functions \longleftrightarrow conservation laws

vector fields \longleftrightarrow symmetries

$T^*(M)$ \longleftrightarrow $\mathcal{L}^*(\mathcal{E})$

$T(M)$ \longleftrightarrow $\mathcal{L}(\mathcal{E})$

De Rham complex \longleftrightarrow $E_1^{0,n-1} \rightarrow E_1^{1,n-1} \rightarrow E_1^{2,n-1} \rightarrow \dots$

multivectors \longleftrightarrow variational multivectors

Schouten bracket \longleftrightarrow variational Schouten bracket

Variational multivectors

Definition

Variational multivectors on \mathcal{E} are conservation laws on $\mathcal{L}^(\mathcal{E})$.*

Theorem

A variational bivector on \mathcal{E} can be identified with the equivalence class of operators A on \mathcal{E} that satisfy the condition

$$l_{\mathcal{E}}A = A^*l_{\mathcal{E}}^*,$$

with two operators being equivalent if they differ by an operator of the form $\square l_{\mathcal{E}}^$.*

If A is a bivector and \mathcal{E} is written in evolution form then $A^* = -A$.

The formula for the Schouten bracket of bivectors

$$\begin{aligned} \llbracket A_1, A_2 \rrbracket(\psi_1, \psi_2) &= \ell_{A_1, \psi_1}(A_2(\psi_2)) - \ell_{A_1, \psi_2}(A_2(\psi_1)) \\ &+ \ell_{A_2, \psi_1}(A_1(\psi_2)) - \ell_{A_2, \psi_2}(A_1(\psi_1)) \\ &\quad - A_1(B_2^*(\psi_1, \psi_2)) - A_2(B_1^*(\psi_1, \psi_2)), \end{aligned}$$

where $\ell_{A, \psi} = \ell_{A(\psi)} - A\ell\psi$,

$\ell_F A_i - A_i^* \ell_F^* = B_i(F, \cdot)$ on J^∞ ,

$B_i^*(\psi_1, \psi_2) = B_i^{*1}(\psi_1, \psi_2)|_{\mathcal{E}}$.

B_i^* are skew-symmetric and skew-adjoint in each argument.

If \mathcal{E} is in evolution form then $B_i^*(\psi_1, \psi_2) = \ell_{A_i, \psi_2}^*(\psi_1)$

Definition

A variational bivectors is called *Hamiltonian* if $\llbracket A, A \rrbracket = 0$

Poisson bracket

$S_1, S_2 \in \text{CL}(\mathcal{E})$, ψ_1, ψ_2 are the generating functions

$$\{S_1, S_2\}_A = E_{A(\psi_1)}(S_2)$$

Definition

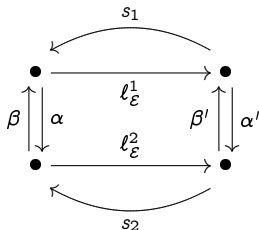
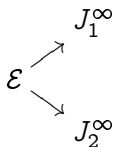
Magri hierarchy on a bihamiltonian equation \mathcal{E} is the infinite sequence S_1, S_2, \dots of conservation laws of \mathcal{E} such that $A_1(\psi_i) = A_2(\psi_{i+1})$.

Proposition

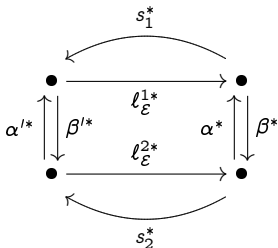
For Magri hierarchy we have

$$\{S_i, S_j\}_{A_1} = \{S_i, S_j\}_{A_2} = \{E_{\varphi_i}, E_{\varphi_j}\} = 0, \text{ where}$$
$$\varphi_i = A_1(\psi_i) = A_2(\psi_{i+1}).$$

Invariance of ℓ^* -covering



$$\ell_\mathcal{E}^1 \beta = \beta' \ell_\mathcal{E}^2, \quad \ell_\mathcal{E}^2 \alpha = \alpha' \ell_\mathcal{E}^1, \quad \beta \alpha = \text{id} + s_1 \ell_\mathcal{E}^1, \quad \alpha \beta = \text{id} + s_2 \ell_\mathcal{E}^2.$$



$$\alpha'^* \beta'^* = \text{id} + s_1^* \ell_\mathcal{E}^{1*}, \quad \beta'^* \alpha'^* = \text{id} + s_2^* \ell_\mathcal{E}^{2*}.$$

Theorem

If $\ell_{\mathcal{E}}^1$ is equivalent to $\ell_{\mathcal{E}}^2$ then $\ell_{\mathcal{E}}^{1*}$ is equivalent to $\ell_{\mathcal{E}}^{2*}$.

Corollary

$\mathcal{L}^*(\mathcal{E})$ doesn't depend on the inclusion $\mathcal{E} \rightarrow J^{\infty}$.

$$A^2 = \alpha A^1 \alpha'^*$$

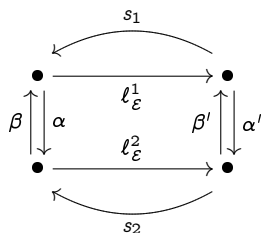
$$A^1 = \beta A^2 \beta'^*$$

Cotangent bundle to a bundle

B. Kupershmidt, *Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalisms*, Lect. Notes Math. 775, 1980, 162–218

$$u_t^1 = 0 \quad u_t^2 = 0 \quad \dots \quad u_t^m = 0$$

Example: KdV



$$F_1 = u_t - u_{xxx} - 6uu_x = 0$$

$$F_2 = \begin{pmatrix} u_x - v \\ v_x - w \\ w_x - u_t + 6uv \end{pmatrix} = 0$$

$$l_{\mathcal{E}}^1 = D_t - D_{xxx} - 6uD_x - 6u_x \quad l_{\mathcal{E}}^2 = \begin{pmatrix} D_x & -1 & 0 \\ 0 & D_x & -1 \\ -D_t + 6v & 6u & D_x \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 1 \\ D_x \\ D_{xx} \end{pmatrix} \quad \alpha' = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \quad \beta = (1 \ 0 \ 0)$$

$$\beta' = (-D_{xx} - 6u \quad -D_x \quad -1)$$

$$s_1 = 0 \quad s_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ D_x & 1 & 0 \end{pmatrix}$$

Example: Camassa-Holm equation

$$u_t - u_{txx} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0$$

$$A_1 = D_x \quad A_2 = -D_t - uD_x + u_x.$$

$$m_t + um_x + 2u_x m = 0,$$

$$m - u + u_{xx} = 0$$

$$A'_1 = \begin{pmatrix} D_x & 0 \\ D_x - D_x^3 & 0 \end{pmatrix} \quad A'_2 = \begin{pmatrix} 0 & -1 \\ 2mD_x + m_x & 0 \end{pmatrix}$$

~~$$u = (1 - D_x^2)^{-1} m$$~~

Example: Kupershmidt deformation

Let \mathcal{E} be a bihamiltonian equation given by $F = 0$

Definition

The Kupershmidt deformation $\tilde{\mathcal{E}}$ has the form

$$F + A_1^*(w) = 0, \quad A_2^*(w) = 0,$$

where $w = (w^1, \dots, w^l)$ are new dependent variables

Theorem

The Kupershmidt deformation $\tilde{\mathcal{E}}$ is a bihamiltonian system.

Proof.

The following two bivectors define a bihamiltonian structures:

$$\tilde{A}_1 = \begin{pmatrix} A_1 & -A_1 \\ 0 & \ell_{F+A_1^*(w)+A_2^*(w)} \end{pmatrix} \quad \tilde{A}_2 = \begin{pmatrix} A_2 & -A_2 \\ -\ell_{F+A_1^*(w)+A_2^*(w)} & 0 \end{pmatrix}$$

□

Magri hierarchy for the Kupershmidt deformation

S_1, S_2, \dots is a Magri hierarchy for \mathcal{E}

ψ_1, ψ_2, \dots are the corresponding generating functions

$$\sum_j D_j(S_j^i) = \langle \psi_i, F \rangle \quad \text{on } J^\infty$$

$$A_1(\psi_i) = A_2(\psi_{i+1}) \quad \text{on } J^\infty$$

Theorem

$(\psi_i, -\psi_{i+1}), i = 1, 2, \dots$ is a Magri hierarchy for the Kupershmidt deformation $\tilde{\mathcal{E}}$