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Hamiltonian structures for general PDEs

Paul Kersten
Joseph Krasil′shchik
Alexander Verbovetsky
Raffaele Vitolo

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Example: KdV

$$
\begin{aligned}
u_t = u_{xxx} + 6uu_x &= D_x\delta(u^3 - u_x^2/2) \\
&= (D_{xxx} + 4uD_x + 2u_x)\delta(u^2/2)
\end{aligned}
$$

$$
u_x = v, \quad v_x = w, \quad w_x = u_t - 6uv
$$

$$
\begin{pmatrix} u \\ v \\ w \end{pmatrix}_x = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -6u \\ 0 & 6u & D_t \end{pmatrix} \delta\left(uw - v^2/2 + 2u^3\right)
$$

$$
\begin{pmatrix} u \\ v \\ w \end{pmatrix}_x = \begin{pmatrix} 0 & -2u & -D_t - 2v \\ 2u & D_t & -12u^2 - 2w \\ -D_t + 2v & 12u^2 + 2w & 8uD_t + 4u_t \end{pmatrix} \delta(-3u^2/2 - w/2)
$$

S. P. Tsarev, *The Hamilton property of stationary and inverse equations of condensed matter mechanics and mathematical physics*, Math. Notes **46** (1989), 569–573

Example: Camassa-Holm equation

$$u_t - u_{txx} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0$$

$$m_t + um_x + 2u_x m = 0, \quad m - u + u_{xx} = 0$$
$$m_t = -um_x - 2u_x m = B_1\,\delta(\mathcal{H}_1) = B_2\,\delta(\mathcal{H}_2)$$

where

$$B_1 = -(mD_x + D_x m), \quad \mathcal{H}_1 = \frac{1}{2}\int mu\,dx$$
$$B_2 = D_x^3 - D_x, \quad \mathcal{H}_2 = \frac{1}{2}\int (u^3 + uu_x^2))\,dx.$$

$\mathcal{H}_1$ and $\mathcal{H}_2$ are viewed as functionals of m and not of u, with $u = (1 - D_x^2)^{-1} m$.

Example: Kupershmidt deformation

B. Kupershmidt, *KdV6: An integrable system*, Phys. Lett. A **372** (2008), 2634–2639

$$u_t = f(t, x, u, u_x, u_{xx}, \dots)$$

A_1, A_2 are compatible Hamiltonian operators
H_1, H_2, ... is a Magri hierarchy of conserved densities
$D_t(H_i) = 0$, $A_1\,\delta(H_i) = A_2\,\delta(H_{i+1})$.

$$u_t = f - A_1(w), \quad A_2(w) = 0 \tag{1}$$

The KdV6 equation
(A. Karasu-Kalkanli, A. Karasu, A. Sakovich, S. Sakovich, and R. Turhan, *A new integrable generalization of the Korteweg-de Vries equation*, arXiv:0708.3247)

$$u_t = u_{xxx} + 6uu_x - w_x, \quad w_{xxx} + 4uw_x + 2u_x w = 0$$

Theorem (Kupershmidt)

H_1, H_2, ... are conserved densities for (1).

Notation: infinite jet space

The jet space J^∞ with coordinates x_i, u^j_σ

$D_i = \partial_{x_i} + \sum_{j,\sigma} u^j_{\sigma i} \partial_{u^j_\sigma}$ are total derivatives

D_i span the Cartan distribution

$E_\varphi = \sum_j \varphi^j \partial_{u^j} + \sum_{ji} D_i(\varphi^j) \partial_{u^j_i} + \dots$ is an evolutionary field,
$\varphi = (\varphi^1, \dots, \varphi^m)$ is a vector function on J^∞

$\ell_f = \left\| \sum_\sigma \partial_{u^j_\sigma}(f_i) D_\sigma \right\|$ is the linearization
of a vector function f on J^∞, $\ell_f(\varphi) = E_\varphi(f)$

$\Delta^* = \| \sum_\sigma (-1)^\sigma D_\sigma a^{ji}_\sigma \|$, if $\Delta = \| \sum_\sigma a^{ij}_\sigma D_\sigma \|$,
the adjoint $\mathcal{C}$-differential operator

Notation: differential equations

Let $F_k(x_i, u^j_\sigma) = 0$, $k = 1, \dots\ l$, be a system of equations

Relations $F = 0$, $D_\sigma(F) = 0$ define its infinite prolongation $\mathcal{E} \subset J^\infty$

$\ell_\mathcal{E} = \ell_F|_\mathcal{E}$ is the linearization of the equation $\mathcal{E}$

E_φ is a symmetry of $\mathcal{E}$ if $E_\varphi(F)|_\mathcal{E} = \ell_\mathcal{E}(\varphi) = 0$, $\mathrm{Sym}(\mathcal{E}) = \ker \ell_\mathcal{E}$
φ is its generating function

Vector function $S = (S^1, \dots, S^n)$ on $\mathcal{E}$ is a conserved current if $\sum_i D_i(S^i) = 0$ on $\mathcal{E}$
A conserved current is trivial if
$S^i = \sum_{j<i} D_j(T^{ji}) - \sum_{i<j} D_j(T^{ij})$
Conservation laws of $\mathcal{E}$ are the conserved currents modulo trivial ones.
Generating function of a conservation law:
$\psi = (\psi_1, \dots, \psi_m) = \Delta^*(1)$, where $\sum_i D_i(S^i) = \Delta(F)$ on J^∞

$$\ell^*_\mathcal{E}(\psi) = 0, \qquad \mathrm{CL}(\mathcal{E}) \subset \ker \ell^*_\mathcal{E}$$

ℓ^*-covering

Cotangent bundle to an equation

the equation $\mathcal{L}^*(\mathcal{E})$ is given by the system

$$\ell_F^*(p) = 0, \quad F = 0$$

ℓ^*-covering:
$$\begin{array}{c} \mathcal{L}^*(\mathcal{E}) \\ \Big\downarrow \tau^* \\ \mathcal{E} \end{array}$$

τ^* is the natural projection $\tau^* \colon (u_\sigma^j, p_\sigma^k) \mapsto (u_\sigma^j)$
variables p_σ^k along the fibers of the covering are odd.

$\langle F, p \rangle$ is the Lagrangian for $\mathcal{L}^*(\mathcal{E})$

Theorem

There is a natural 1-1 *correspondence between the symmetries of* $\mathcal{E}$ *and the conservation laws of* $\mathcal{L}^*(\mathcal{E})$ *linear along the fibers of* τ^*.

φ is a symmetry $\Rightarrow \ell_F(\varphi) = \Delta(F)$, φ_Δ corresponds to Δ^*
$(\varphi, \varphi_\Delta)$ is the conservation law

Dictionary

Manifold M		**PDE** $\mathcal{E}$
functions	$\longleftrightarrow$	conservation laws
vector fields	$\longleftrightarrow$	symmetries
$T^*(M)$	$\longleftrightarrow$	$\mathcal{L}^*(\mathcal{E})$
$T(M)$	$\longleftrightarrow$	$\mathcal{L}(\mathcal{E})$
De Rham complex	$\longleftrightarrow$	$E_1^{0,n-1} \to E_1^{1,n-1} \to E_1^{2,n-1} \to \cdots$
multivectors	$\longleftrightarrow$	variational multivectors
Schouten bracket	$\longleftrightarrow$	variational Schouten bracket

Variational multivectors

Definition

Variational multivectors on $\mathcal{E}$ are conservation laws on $\mathcal{L}^*(\mathcal{E})$.

Theorem

A variational bivector on $\mathcal{E}$ can be identified with the equivalence class of operators A on $\mathcal{E}$ that satisfy the condition

$$\ell_{\mathcal{E}} A = A^* \ell_{\mathcal{E}}^*,$$

with two operators being equivalent if they differ by an operator of the form $\square\, \ell_{\mathcal{E}}^*$.

If A is a bivector and $\mathcal{E}$ is written in evolution form then $A^* = -A$.

The formula for the Schouten bracket of bivectors

$$\begin{aligned}
&[\![A_1, A_2]\!](\psi_1, \psi_2) \\
&\qquad = \ell_{A_1,\psi_1}(A_2(\psi_2)) - \ell_{A_1,\psi_2}(A_2(\psi_1)) \\
&\qquad + \ell_{A_2,\psi_1}(A_1(\psi_2)) - \ell_{A_2,\psi_2}(A_1(\psi_1)) \\
&\qquad\qquad - A_1(B_2^*(\psi_1, \psi_2)) - A_2(B_1^*(\psi_1, \psi_2)),
\end{aligned}$$

where $\ell_{A,\psi} = \ell_{A(\psi)} - A\ell_\psi$,

$\ell_F A_i - A_i^* \ell_F^* = B_i(F, \cdot)$ on J^∞,

$B_i^*(\psi_1, \psi_2) = B_i^{*_1}(\psi_1, \psi_2)|_{\mathcal{E}}$.

B_i^* are skew-symmetric and skew-adjoint in each argument.

If $\mathcal{E}$ is in evolution form then $B_i^*(\psi_1, \psi_2) = \ell^*_{A_i,\psi_2}(\psi_1)$

Definition

A variational bivectors is called *Hamiltonian* if $[\![A, A]\!] = 0$

Poisson bracket

$S_1, S_2 \in \mathrm{CL}(\mathcal{E})$, ψ_1, ψ_2 are the generating functions

$\{S_1, S_2\}_A = E_{A(\psi_1)}(S_2)$

Definition

Magri hierarchy on a bihamiltonian equation $\mathcal{E}$ is the infinite sequence S_1, S_2, ... of conservation laws of $\mathcal{E}$ such that $A_1(\psi_i) = A_2(\psi_{i+1})$.

Proposition

For Magri hierarchy we have $\{S_i, S_j\}_{A_1} = \{S_i, S_j\}_{A_2} = \{E_{\varphi_i}, E_{\varphi_j}\} = 0,$ *where* $\varphi_i = A_1(\psi_i) = A_2(\psi_{i+1})$.

Invariance of ℓ^*-covering

$$
\begin{tikzcd}
 & J_1^\infty \\
\mathcal{E} \arrow[ur] \arrow[dr] & \\
 & J_2^\infty
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=huge, row sep=large]
\bullet \arrow[r, "\ell_{\mathcal{E}}^1"'] \arrow[d, shift left, "\alpha"] & \bullet \arrow[l, bend right, "s_1"'] \arrow[d, shift left, "\alpha'"] \\
\bullet \arrow[r, "\ell_{\mathcal{E}}^2"] \arrow[u, shift left, "\beta"] & \bullet \arrow[l, bend left, "s_2"] \arrow[u, shift left, "\beta'"]
\end{tikzcd}
$$

$$\ell_{\mathcal{E}}^1\,\beta = \beta'\,\ell_{\mathcal{E}}^2, \quad \ell_{\mathcal{E}}^2\,\alpha = \alpha'\,\ell_{\mathcal{E}}^1, \qquad \beta\,\alpha = \mathrm{id}+s_1\,\ell_{\mathcal{E}}^1, \quad \alpha\,\beta = \mathrm{id}+s_2\,\ell_{\mathcal{E}}^2.$$

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\bullet \arrow[r, "\ell_{\mathcal{E}}^{1*}"'] \arrow[d, shift left, "\beta'^*"] & \bullet \arrow[l, bend right, "s_1^*"'] \arrow[d, shift left, "\beta^*"] \\
\bullet \arrow[r, "\ell_{\mathcal{E}}^{2*}"] \arrow[u, shift left, "\alpha'^*"] & \bullet \arrow[l, bend left, "s_2^*"] \arrow[u, shift left, "\alpha^*"]
\end{tikzcd}
$$

$$\alpha'^*\,\beta'^* = \mathrm{id} + s_1^*\,\ell_{\mathcal{E}}^{1*}, \quad \beta'^*\,\alpha'^* = \mathrm{id} + s_2^*\,\ell_{\mathcal{E}}^{2*}.$$

Theorem

If $\ell^1_{\mathcal{E}}$ is equivalent to $\ell^2_{\mathcal{E}}$ then $\ell^{1*}_{\mathcal{E}}$ is equivalent to $\ell^{2*}_{\mathcal{E}}$.

Corollary

$\mathcal{L}^*(\mathcal{E})$ doesn't depend on the inclusion $\mathcal{E} \to J^\infty$.

$$A^2 = \alpha\, A^1\, \alpha'^*$$
$$A^1 = \beta\, A^2\, \beta'^*$$

Cotangent bundle to a bundle

B. Kupershmidt, *Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalisms*, Lect. Notes Math. 775, 1980, 162–218

$$u^1_t = 0 \quad u^2_t = 0 \quad \dots \quad u^m_t = 0$$

Example: KdV

$$
\begin{tikzcd}[row sep=large, column sep=huge]
\bullet \arrow[r, "\ell^1_{\mathcal{E}}"'] \arrow[d, shift left, "\alpha"] & \bullet \arrow[l, bend right, "s_1"'] \arrow[d, shift left, "\alpha'"] \\
\bullet \arrow[r, "\ell^2_{\mathcal{E}}"] \arrow[u, shift left, "\beta"] & \bullet \arrow[l, bend left, "s_2"] \arrow[u, shift left, "\beta'"]
\end{tikzcd}
$$

$$F_1 = u_t - u_{xxx} - 6uu_x = 0$$

$$F_2 = \begin{pmatrix} u_x - v \\ v_x - w \\ w_x - u_t + 6uv \end{pmatrix} = 0$$

$$\ell^1_{\mathcal{E}} = D_t - D_{xxx} - 6uD_x - 6u_x \qquad \ell^2_{\mathcal{E}} = \begin{pmatrix} D_x & -1 & 0 \\ 0 & D_x & -1 \\ -D_t + 6v & 6u & D_x \end{pmatrix}$$

$$\alpha = \begin{pmatrix} 1 \\ D_x \\ D_{xx} \end{pmatrix} \quad \alpha' = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

$$\beta' = \begin{pmatrix} -D_{xx} - 6u & -D_x & -1 \end{pmatrix}$$

$$s_1 = 0 \quad s_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ D_x & 1 & 0 \end{pmatrix}$$

Example: Camassa-Holm equation

$$u_t - u_{txx} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0$$

$$A_1 = D_x \qquad A_2 = -D_t - uD_x + u_x.$$

$$\begin{aligned} & m_t + um_x + 2u_x m = 0, \\ & m - u + u_{xx} = 0 \end{aligned}$$

$$A'_1 = \begin{pmatrix} D_x & 0 \\ D_x - D_x^3 & 0 \end{pmatrix} \qquad A'_2 = \begin{pmatrix} 0 & -1 \\ 2mD_x + m_x & 0 \end{pmatrix}$$

$$\xcancel{u = (1 - D_x^2)^{-1} m}$$

Example: Kupershmidt deformation

Let $\mathcal{E}$ be a bihamiltonian equation given by $F = 0$

Definition

The Kupershmidt deformation $\tilde{\mathcal{E}}$ has the form

$$F + A_1^*(w) = 0, \qquad A_2^*(w) = 0,$$

where $w = (w^1, \dots, w^l)$ are new dependent variables

Theorem

The Kupershmidt deformation $\tilde{\mathcal{E}}$ is a bihamiltonian system.

Proof.

The following two bivectors define a bihamiltonian structures:

$$\tilde{A}_1 = \begin{pmatrix} A_1 & -A_1 \\ 0 & \ell_{F+A_1^*(w)+A_2^*(w)} \end{pmatrix} \quad \tilde{A}_2 = \begin{pmatrix} A_2 & -A_2 \\ -\ell_{F+A_1^*(w)+A_2^*(w)} & 0 \end{pmatrix}$$

$\square$

Magri hierarchy for the Kupershmidt deformation

$S_1, S_2, \ldots$ is a Magri hierarchy for $\mathcal{E}$

$\psi_1, \psi_2, \ldots$ are the corresponding generating functions

$$\sum_j D_j(S_j^i) = \langle \psi_i, F \rangle \quad \text{on } J^\infty$$

$$A_1(\psi_i) = A_2(\psi_{i+1}) \quad \text{on } J^\infty$$

Theorem

$(\psi_i, -\psi_{i+1})$, $i = 1, 2, \ldots$ is a Magri hierarchy for the Kupershmidt deformation $\tilde{\mathcal{E}}$