Internal Lagrangians as variational principles

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Geometry of Differential Equations, IUM
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Outline

1. ## Main problems
   - From a cohomological point of view, it seems reasonable to say that variational equations encode their variational nature through certain cohomology elements, which we call internal Lagrangians. But what is the meaning of these cohomologies? Why (how) do they encode admissible Lagrangians?
   - What details about the variational nature of a differential equation are known to its internal Lagrangians?

2. ## Main results
   - Each principle of stationary action reproduces itself in terms of the intrinsic geometry of a variational equation.
   - One can consider variations of internal Lagrangians within different classes of submanifolds. Non-degenerate Lagrangians produce internal Lagrangians, which can be considered non-degenerate in some sense.

Basic notation

Let us consider a locally trivial smooth vector bundle $\pi\colon E \to M$. Here

- $\dim M = n$, $\dim E = n + m$;
- $x^1, \ldots, x^n$ are local coordinates in $U \subset M$ (independent variables);
- $u^1, \ldots, u^m$ are local coordinates along the fibres of π over U.

The bundle π determines the corresponding bundle of infinite jets

$$\pi_\infty\colon J^\infty(\pi) \to M$$

with the adapted local coordinates u^i_α along the fibers. The Cartan (Pfaff, Lie, contact) distribution is spanned by the total derivatives

$$D_{x^k} = \partial_{x^k} + u^i_{\alpha+x^k}\partial_{u^i_\alpha}\,, \qquad k = 1, \ldots, n, \ |\alpha| \geqslant 0.$$

Here α is a multi-index of the form $\alpha = \alpha_i x^i$ (just a formal linear combination), $\alpha_i \geqslant 0$; $\quad |\alpha| = \alpha_1 + \ldots + \alpha_n$.

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By a Cartan differential form we mean a form vanishing on the Cartan distribution. A Cartan 1-form $\omega \in \mathcal{C}\Lambda^1(\pi)$ can be written as a finite sum

$$\omega = \omega_i^\alpha \theta_\alpha^i\,, \qquad \theta_\alpha^i = du_\alpha^i - u_{\alpha+x^k}^i dx^k \tag{1}$$

in adapted local coordinates. The module $\mathcal{C}\Lambda^1(\pi)$ determines the corresponding ideal of the algebra $\Lambda^*(\pi)$. We denote by $\mathcal{C}^2\Lambda^*(\pi)$ the wedge square of this ideal.

Horizontal n-forms can be regarded as Lagrangians

$$\Lambda_h^n(\pi) = \Lambda^n(\pi)/\mathcal{C}\Lambda^n(\pi)\,. \tag{2}$$

If $L \in \Lambda^n(\pi)$ has the form $L = \lambda\, dx^1 \wedge \ldots \wedge dx^n$, then $\mathrm{E}[L]_h$ is defined by

$$\mathrm{E}[L]_h = (-1)^{|\alpha|} D_\alpha\Big(\frac{\partial\lambda}{\partial u_\alpha^i}\Big)\, \theta_0^i \wedge dx^1 \wedge \ldots \wedge dx^n. \tag{3}$$

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Suppose we have a system of differential equations

$$F = 0\,, \tag{4}$$

where F is a section of some bundle of the form $\pi_\infty^*(\eta)$. Denote by $\mathcal{E}$ its infinite prolongation. If $L \in \Lambda^n(\pi)$, then a certain cohomological reasoning (Noether's identity $\mathcal{L}_{E_\varphi}[L]_h = i_{E_\varphi}\mathrm{E}[L]_h + d_h[i_{E_\varphi}\omega_L]$, the Vinogradov $\mathcal{C}$-spectral sequence) leads to various decompositions of the form

$$dL - \mathrm{E}[L]_h \in \mathcal{C}^2\Lambda^{n+1}(\pi) + d(\mathcal{C}\Lambda^n(\pi))\,. \tag{5}$$

This results in the following commutative diagram

$$[L]_h \ \text{ such that } \ \mathrm{E}[L]_h|_{\mathcal{E}} = 0 \longrightarrow [L]_h + d_h\Lambda_h^{n-1}(\pi)$$

$$S \in \frac{\{l \in \Lambda^n(\mathcal{E})\colon\ dl \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{\mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\mathcal{C}\Lambda^{n-1}(\mathcal{E}))} \longrightarrow \ell \in \frac{\{l \in \Lambda^n(\mathcal{E})\colon\ dl \in \mathcal{C}^2\Lambda^{n+1}(\mathcal{E})\}}{\mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\Lambda^{n-1}(\mathcal{E}))}$$

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Integral functionals

To obtain a uniquely defined integral functional, it is necessary to get rid of

- terms that belong to $\mathcal{C}^2\Lambda^n(\mathcal{E})$,
- the boundary terms $d(\mathcal{C}\Lambda^{n-1}(\mathcal{E}))$.

Further we prefer the Lagrangian approach to the Eulerian one and consider deformations of embeddings rather than motions of their images.

Let us fix a compact oriented n-dimensional manifold N with boundary.

Definition

An embedding $\sigma\colon N \to \mathcal{E}$ is an *almost Cartan embedding* if

$$\dim\left(T_p\,\sigma(N) \cap \mathcal{C}_p\right) \geqslant n-1 \qquad \text{for all} \quad p \in \sigma(N)\,.$$

Notation: $\sigma \in \mathcal{A}_N(\mathcal{E})$.

Motivation: if $\sigma \in \mathcal{A}_N(\mathcal{E})$, then $\sigma^*(\mathcal{C}^2\Lambda^n(\mathcal{E})) = 0$.

Example

Consider the heat equation $u_t = u_{xx}$ and its infinite prolongation

$$\mathcal{E}: \qquad u_t - u_{xx} = 0, \quad D_x(u_t - u_{xx}) = 0, \quad D_t(u_t - u_{xx}) = 0, \quad \ldots$$

We can regard x, t, u and all the derivatives w.r.t. x as coordinates on $\mathcal{E}$. Suppose we have some IVP : $t = 0,\ u = f_0(x)$. All the coordinate functions $u_x,\ u_{xx},\ \ldots$ can be determined using these data and $\bar{D}_x = D_x|_{\mathcal{E}}$:

$$u = f_0(x), \quad u_x = \partial_x f_0(x), \quad u_{xx} = \partial_x^2 f_0(x), \quad u_{xxx} = \partial_x^3 f_0(x), \qquad \ldots$$

We can consider IVPs of the form $u = f(x, t_0)$ for all $t_0 \in \mathbb{R}$ and apply the same approach to them:

$$u = f(x, t_0), \qquad u_x = \partial_x f(x, t_0), \qquad u_{xx} = \partial_x^2 f(x, t_0), \qquad \ldots$$

Substituting t for t_0 in these formulas, we obtain an embedding of $\mathbb{R}^2$ to $\mathcal{E}$. The restriction of this embedding to a compact submanifold of $\mathbb{R}^2$ is an almost Cartan embedding, since its image is tangent to the vector field $\bar{D}_x$.

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A $\sigma \in \mathcal{A}_N(\mathcal{E})$ *defines a boundary value problem* if

$$T_p\,\sigma(\partial N) \subset \mathcal{C}_p \qquad \text{for all} \quad p \in \sigma(\partial N)\,.$$

Notation: $\sigma \in \mathcal{BA}_N(\mathcal{E})$.

(This is how solutions behave on ∂N). Motivation: if $\sigma \in \mathcal{BA}_N(\mathcal{E})$, then

$$\int_N \sigma^*(\mathcal{C}^2\Lambda^n(\mathcal{E}) + d\mathcal{C}\Lambda^{n-1}(\mathcal{E})) = \int_{\partial N} \sigma^*(\mathcal{C}\Lambda^{n-1}(\mathcal{E})) = 0. \qquad (6)$$

Let $[L]_h$ be a horizontal n-form such that the variational derivative $\mathrm{E}[L]_h$ vanishes on $\mathcal{E}$. Then $[L]_h$ determines a unique integral functional

$$S\colon \mathcal{BA}_N(\mathcal{E}) \to \mathbb{R}\,, \qquad S(\sigma) = \int_N \sigma^*(l)\,, \qquad (7)$$

where l represents an element of the corresponding quotient. If σ defines a solution to $\mathcal{E}$, then $S(\sigma)$ coincide with the value of the original action on σ.

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Current picture

A compact oriented n-dimensional manifold N, and

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\text{functional on } \mathcal{BA}_N(\mathcal{E}) & & \text{ambiguous functional on } \mathcal{A}_N(\mathcal{E})
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However, we prefer to define stationary points of internal Lagrangians, since this approach allows us to avoid some technical difficulties. Therefore, below we take into account all almost Cartan embeddings $\mathcal{A}_N(\mathcal{E})$, and not only those that define boundary value problems.

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Variation of an internal Lagrangian

Consider a path in $\mathcal{A}_N(\mathcal{E})$, that is, a smooth mapping $\gamma\colon \mathbb{R}\times N \to \mathcal{E}$ such that for all $\tau\in\mathbb{R}$ the mappings

$$\gamma(\tau)\colon N\to\mathcal{E}\,, \qquad \gamma(\tau)\colon x\mapsto\gamma(\tau,x) \tag{8}$$

are almost Cartan embeddings. Let 0_N denote the zero-section

$$0_N\colon N\to\mathbb{R}\times N, \qquad 0_N(x)=(0,x). \tag{9}$$

Then $\gamma(0)=\gamma\circ 0_N$. If the boundary is fixed, we obtain

$$\frac{d}{d\tau}\Big|_{\tau=0}\int_N \gamma(\tau)^*(l) = \int_N 0_N^*\big(i_{\partial_\tau}\gamma^*(dl)\big). \tag{10}$$

So, the derivative along a path γ is completely determined by the corresponding presymplectic structure $dl + \mathcal{C}^3\Lambda^{n+1}(\mathcal{E}) + d(\mathcal{C}^2\Lambda^n(\mathcal{E}))$.

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functional on $\mathcal{BA}_N(\mathcal{E})$ — ambiguous functional on $\mathcal{A}_N(\mathcal{E})$

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unambiguous variations along paths in $\mathcal{BA}_N(\mathcal{E})$ — unambiguous variations along paths in $\mathcal{A}_N(\mathcal{E})$

(due to the corresponding presymplectic structure)

Definition

A $\sigma \in \mathcal{A}_N(\mathcal{E})$ is a *stationary point* of $\ell = l + \mathcal{C}^2\Lambda^n(\mathcal{E}) + d(\Lambda^{n-1}(\mathcal{E}))$ if

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(l) = 0 \qquad (11)$$

for any path γ in $\mathcal{A}_N(\mathcal{E})$ such that $\gamma(0) = \sigma$ and the boundary is fixed.

If σ defines a solution to $\mathcal{E}$, then it is a stationary point of any internal Lagrangian of $\mathcal{E}$:

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(l) = \int_N 0_N^*\left(i_{\partial_\tau}\gamma^*(dl)\right) \qquad (12)$$

One can also define stationary points of conservation laws the same way.

Stationary points of internal Lagrangians

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Example

Consider Laplace's equation $u_{yy} = -u_{xx}$ and its infinite prolongation $\mathcal{E}$. Let $N \subset \mathbb{R}^2$ be a compact submanifold. There is the Lagrangian $[L]_h$, where

$$L = -\frac{u_x^2 + u_y^2}{2}\, dx \wedge dy\,. \tag{13}$$

The corresponding internal Lagrangian is represented by the restriction of

$$L + \omega_L = -\frac{u_x^2 + u_y^2}{2}\, dx \wedge dy - u_x\, \theta_0 \wedge dy + u_y\, \theta_0 \wedge dx\,. \tag{14}$$

Suppose $Y \in C^\infty(N)$ is a function on N, and $\sigma \in \mathcal{A}_N(\mathcal{E})$ is a (local) section of the bundle $\pi_\infty|_\mathcal{E}$ such that

$$d\sigma(\partial_x + Y\partial_y) = \bar{D}_x + Y\bar{D}_y \qquad \text{for all} \quad (x, y) \in N. \tag{15}$$

Then in local coordinates on $\mathcal{E}$ we have

$$\sigma: \qquad u = f\,, \qquad u_x = \partial_x f + Y(\partial_y f - g), \qquad u_y = g\,, \qquad \ldots \tag{16}$$

Arbitrary smooth functions $f, g\colon N \to \mathbb{R}$ determine an appropriate σ and vice versa.

Example

Consider Laplace's equation $u_{yy} = -u_{xx}$ and its infinite prolongation $\mathcal{E}$. Let $N \subset \mathbb{R}^2$ be a compact submanifold. There is the Lagrangian $[L]_h$, where

$$L = -\frac{u_x^2 + u_y^2}{2}\, dx \wedge dy\,. \tag{13}$$

The corresponding internal Lagrangian is represented by the restriction of

$$L + \omega_L = -\frac{u_x^2 + u_y^2}{2}\, dx \wedge dy - u_x\, \theta_0 \wedge dy + u_y\, \theta_0 \wedge dx\,. \tag{14}$$

Suppose $Y \in C^\infty(N)$ is a function on N, and $\sigma \in \mathcal{A}_N(\mathcal{E})$ is a (local) section of the bundle $\pi_\infty|_{\mathcal{E}}$ such that

$$d\sigma(\partial_x + Y\partial_y) = \bar{D}_x + Y\bar{D}_y \qquad \text{for all} \quad (x, y) \in N. \tag{15}$$

Then in local coordinates on $\mathcal{E}$ we have

$$\sigma\colon \qquad u = f\,, \qquad u_x = \partial_x f + Y(\partial_y f - g), \qquad u_y = g\,, \qquad \ldots \tag{16}$$

Arbitrary smooth functions $f, g\colon N \to \mathbb{R}$ determine an appropriate σ and vice versa.

Example

Let δf, δg, and δY be arbitrary smooth functions on N vanishing together with all their derivatives on ∂N. Substituting $f + \tau\delta f$, $g + \tau\delta g$, and $Y + \tau\delta Y$ for f, g, and Y in σ, we obtain the corresponding path γ. It suffices to consider the pullback

$$\gamma(0)^*(l) = \Big(\frac{Y^2(\partial_y f - g)^2 - (\partial_x f)^2 + g^2}{2} - g\partial_y f\Big) dx \wedge dy\,, \tag{17}$$

where $l = (L + \omega_L)|_{\mathcal{E}}$. Varying w.r.t. the variables f and g, we obtain

$$\partial_x^2 f + \partial_y g + \partial_y\big(Y^2(g - \partial_y f)\big) = 0\,, \quad (g - \partial_y f)(Y^2 + 1) = 0\,. \tag{18}$$

So, here we don't even need to vary w.r.t. the variable Y. As we will see below, this is not a coincidence.

Thus, an almost Cartan embedding $\sigma \in \mathcal{A}_N(\mathcal{E})$ that is a local section of the bundle $\pi_\infty|_{\mathcal{E}}$ is a stationary point of the internal Lagrangian under consideration iff it defines a solution to $\mathcal{E}$.

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Infinitesimal symmetries of an infinitely prolonged system of equations $\mathcal{E}$ act on its internal Lagrangians by means of the Lie derivative.

The de Rham differential d induces the mapping from internal Lagrangians to presymplectic structures, i.e., elements of the kernel of the differential

$$d_1^{2,\,n-1}\colon E_1^{2,\,n-1}(\mathcal{E}) \to E_1^{3,\,n-1}(\mathcal{E})\,. \tag{19}$$

Here $E_r^{p,\,q}(\mathcal{E})$ are groups of the Vinogradov $\mathcal{C}$-spectral sequence. The inclusions

$$\Lambda^*(\mathcal{E}) \supset \mathcal{C}\Lambda^*(\mathcal{E}) \supset \mathcal{C}^2\Lambda^*(\mathcal{E}) \tag{20}$$

allow us to establish the following version of the Noether theorem

Theorem

Let ℓ be an internal Lagrangian, and let X be an infinitesimal symmetry of an infinitely prolonged system of differential equations $\mathcal{E}$. If ℓ is invariant under the action of X, then X gives rise to conservation laws. Otherwise, X produces the non-trivial internal Lagrangian $\mathcal{L}_X\,\ell$.

Besides, if $\mathcal{E}$ admits gauge symmetries, then all its internal Lagrangians are gauge invariant (since all its presymplectic structures are).

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ξ-stationary points

Suppose $N \subset M^n$ is a compact oriented n-dimensional submanifold, where M is the base of some jets bundle π_∞. Denote the restriction $\pi_\infty|_\mathcal{E}$ by $\pi_\mathcal{E}$.

Definition

We say that $\sigma \in \mathcal{A}_N(\mathcal{E})$ is an *almost Cartan section* of $\pi_\mathcal{E}$ if $\pi_\mathcal{E} \circ \sigma = \mathrm{id}_N$.

Definition

Let $\xi \in \Lambda^1(M)$ be a covector field. An almost Cartan section $\sigma\colon N \to \mathcal{E}$ is a *ξ-section* of $\pi_\mathcal{E}$ if $d\sigma(\ker \xi|_x) \subset \mathcal{C}_{\sigma(x)}$ for all $x \in N$.

Definition

Let ℓ be an internal Lagrangian of $\mathcal{E}$, $l \in \ell$ be a differential form representing ℓ. We say that a ξ-section σ is a *ξ-stationary point* of ℓ if

$$\frac{d}{d\tau}\Big|_{\tau=0} \int_N \gamma(\tau)^*(l) = 0$$

for any path in ξ-sections such that $\gamma(0) = \sigma$ and the boundary is fixed.

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On non-degenerate Lagrangians

If a ξ-section is a stationary point of an internal Lagrangian, it is also a ξ-stationary one.

Theorem

Let L be a differential n-form on $J^k(\pi)$, $k \geqslant 1$, and let $\mathcal{E}$ be the infinite prolongation of the Euler-Lagrange equations $\mathrm{E}[L]_h = 0$. Suppose $\xi \in \Lambda^1(M)$ is a non-vanishing, non-characteristic covector field for the $\{\mathrm{E}[L]_h = 0\} \subset J^{2k}(\pi)$ such that the distribution $\xi = 0$ is integrable. Then a ξ-section is a ξ-stationary point of the corresponding internal Lagrangian if and only if it is a (local) solution to $\pi_{\mathcal{E}}$.

One can say that such ξ is related to $(n-1, 1)$ decomposition. This theorem is inapplicable to gauge theories. However, it is worth noting that internal Lagrangians can be restricted to subsystems that arise after gauge fixing.

Let us also note that the Proca theory and the massive spin-2 theory are examples of degenerate non-gauge Euler-Lagrange expressions.

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Let us also note that the Proca theory and the massive spin-2 theory are examples of degenerate non-gauge Euler-Lagrange expressions.

Characteristics and internal Lagrangians

Let us denote by $\mathcal{E}$ the infinite prolongation of the wave equation

$$u_{xy} = 0\,.$$

Suppose $N \subset \mathbb{R}^2$ is a connected compact 2-dimensional submanifold. Any dy-section $N \to \mathcal{E}$ has the form

$$u = f\,, \quad u_x = \partial_x f\,, \quad u_y = h_1, \quad u_{xx} = \partial_x^2 f\,, \quad u_{yy} = h_2, \quad \ldots \tag{21}$$

Here $f, h_1, h_2, \ldots$ are functions on N. The function f can be chosen arbitrarily, while the functions h_1, h_2, ... do not depend on the variable x.

But using the internal Lagrangian of the wave equation, we cannot get an infinite number of relations between these functions h_1, h_2, h_3, ...

So, characteristics can cause problems.

Characteristics and internal Lagrangians

Consider the 1-d Schrödinger equation for $\hbar = 1$, $m = 1/2$, $\Psi = u + iv$:

$$-v_t + u_{xx} - V(x)u = 0\,, \qquad u_t + v_{xx} - V(x)v = 0\,. \tag{22}$$

This system is a Lagrangian one,

$$L = \Big(\frac{u_t v - u v_t}{2} - \frac{u_x^2 + v_x^2}{2} - V(x)\,\frac{u^2 + v^2}{2}\Big) dt \wedge dx\,. \tag{23}$$

Suppose $N \subset \mathbb{R}^2$ is a connected compact 2-dimensional submanifold. The covector field dt defines the characteristic subdistribution.

Choosing u_t, v_t as external coordinates for the infinite prolongation $\mathcal{E}$, we see that any dt-section $\sigma\colon N \to \mathcal{E}$ has the form

$$\sigma\colon \qquad u = f\,, \qquad v = g\,, \qquad u_x = \partial_x f\,, \qquad v_x = \partial_x g\,, \qquad \ldots \tag{24}$$

The functions f and g are arbitrary functions on N.

The desired internal Lagrangian is represented by the form

$$l = -\frac{u_x^2 + uu_{xx} + v_x^2 + vv_{xx}}{2}\, dt \wedge dx + \frac{1}{2}(v\,\theta_0^u - u\,\theta_0^v) \wedge dx - \\ - dt \wedge (u_x\,\theta_0^u + v_x\,\theta_0^v). \qquad (25)$$

Assume that $\gamma\colon \mathbb{R} \times N \to \mathcal{E}$ is a path in dt-sections such that the boundary is fixed. Then we can only vary w.r.t. the variables f and g. Eventually, we find

$$\sigma^*(l) = \Big(\frac{g\partial_t f - f\partial_t g}{2} - \frac{(\partial_x f)^2 + (\partial_x g)^2}{2} - V(x)\,\frac{f^2 + g^2}{2}\Big) dt \wedge dx. \quad (26)$$

Comparing this pullback with the original Lagrangian, we can conclude that all dt-stationary points of the internal Lagrangian are local solutions to $\pi_{\mathcal{E}}$.

So, one can identify dt-sections of an evolutionary system of equations with sections of its original bundle $\pi\colon E \to M$. In this case, the variation of an internal Lagrangian within the class of dt-sections has the same physical meaning as the variation of the corresponding Lagrangian (on jets).

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A surprising example

Consider the potential KdV equation

$$u_t - 3u_x^2 - u_{xxx} = 0 \tag{27}$$

and its infinite prolongation $\mathcal{E}$. This equation admits the presymplectic operator $\Delta = \bar{D}_x$ with non-trivial kernel. The corresponding presymplectic structure should be considered a degenerate one. It can be related only to a variational principle that gives a consequence of the original equation, but not the potential KdV itself.

There exists a unique internal Lagrangian ℓ producing the same presymplectic structure:

$$l = \Big(\frac{u_x u_t}{2} - u_x^3 + \frac{u_{xx}^2}{2}\Big) dt \wedge dx - \frac{1}{2} u_t\, dt \wedge \theta_0 + u_{xx}\, dt \wedge \theta_x + \frac{1}{2} u_x\, \theta_0 \wedge dx\,,$$

where $\theta_0 = du - u_x\, dx - u_t\, dt$ and $\theta_x = du_x - u_{xx}\, dx - u_{xt}\, dt$. Here we regard the variable u_{xxx} and its derivatives as external coordinates for $\mathcal{E}$.

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A surprising example

Suppose $N \subset \mathbb{R}^2$ is a connected compact 2-dimensional submanifold. Let us define ξ by

$$\xi = dx - X(t, x)dt \,. \tag{28}$$

Then a ξ-section σ is of the form

$$\sigma : \qquad \begin{array}{lll} u = f, & u_x = g, & u_t = \partial_t f + X(\partial_x f - g), \\ u_{xx} = h, & u_{xt} = \partial_t g + X(\partial_x g - h), & \dots \end{array} \tag{29}$$

Here f, g and h are arbitrary functions on N. The expressions for all other coordinates on $\mathcal{E}$ are unambiguously defined.

So, we get the pullback

$$\sigma^*(l) = \Big(-\frac{\partial_x f \, \partial_t f}{2} + g \, \partial_t f - g^3 + h \, \partial_x g - \frac{h^2}{2} - \frac{X}{2}(\partial_x f - g)^2 \Big) dt \wedge dx \,.$$

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Varying the corresponding action with respect to f, g and h, we obtain

$$\begin{aligned} &\partial_t(\partial_x f - g) + \partial_x\big(X(\partial_x f - g)\big) = 0\,,\\ &\partial_t f - 3g^2 - \partial_x h + X(\partial_x f - g) = 0\,,\\ &\partial_x g - h = 0\,. \end{aligned} \tag{30}$$

These equations do not imply the relation $\partial_x f = g$. Therefore, the set of ξ-stationary points contains more than just local solutions. However, we can also vary w.r.t. X (i.e., perturb ξ). As a result, we get the missing

$$\partial_x f = g\,. \tag{31}$$

Thus, if ξ determines a non-characteristic distribution, then a ξ-section σ is a stationary point of the internal Lagrangian ℓ iff σ is a local solution. But this is not the case for ξ-stationary points.

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