Hierarchies of pseudo difference operators and their Darboux transformations

Joint with
A.G.Helminck, A. Opimakh, L.Panasenko,
V. Poberezhny, S.Polenkova

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Outline of the talk

- Hierarchies
- The Lie algebra PsΔ
- Compatible Lax equations
- Related Cauchy problems
- Geometric construction of solutions
- Darboux transformations

Hierarchies 1

- General set-up for hierarchies: Lie algebra $\mathfrak{g}$
- $\mathfrak{g}_i, i = 1, 2$, Lie subalgebras of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

- π_i the projection of $\mathfrak{g}$ onto $\mathfrak{g}_i$ induced by this decomposition
- $\mathfrak{g}_2$ Lie algebra of the Lie subgroup G_2
- Set linear independent, commuting elements:

$$\{F_j \mid j \geq 1\} \in \mathfrak{g}_1$$

- t_j flow parameter w.r.t. F_j, $\partial_j = \frac{\partial}{\partial t_j}$, $t = \{t_j\}$.

Hierarchies 2

- Search for $g_2(t) \in G_2$ such that the deformations

$$\mathcal{F}_j := g_2(t)^{-1} F_j g_2(t), j \geq 1$$

satisfy for all $j_1 \geqslant 1$ and $j_2 \geqslant 1$:

$$\frac{\partial}{\partial t_{j_1}}(\mathcal{F}_{j_2}) = [\mathcal{F}_{j_2}, \pi_2(\mathcal{F}_{j_1})] = [\pi_1(\mathcal{F}_{j_1}), \mathcal{F}_{j_2}] \tag{1}$$

- The last equality in (1) follows from $[\mathcal{F}_{j_1}, \mathcal{F}_{j_2}] = 0$.
- (1): *compatible Lax equations*, for in practice it implies

$$\frac{\partial}{\partial t_{j_1}}(\pi_1(\mathcal{F}_{j_2})) - \frac{\partial}{\partial t_{j_2}}(\pi_1(\mathcal{F}_{j_1})) - [\pi_1(\mathcal{F}_{j_1}), \pi_1(\mathcal{F}_{j_2})] = 0,$$

a set of *zero curvature relations*.

Pseudo difference operators 1

- Commutative k-algebra R, $k = \mathbb{R}$ or $\mathbb{C}$.
- $M_{\mathbb{Z}}(R)$: $\mathbb{Z} \times \mathbb{Z}$-matrices, coefficients from R
- $A = (a_{ij}) \in M_{\mathbb{Z}}(R)$:

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{a_{n-1\ n-1}} & a_{n-1\ n} & a_{n-1\ n+1} & \ddots \\ \ddots & a_{n\ n-1} & \mathbf{a_{n\ n}} & a_{n\ n+1} & \ddots \\ \ddots & a_{n+1\ n-1} & a_{n+1\ n} & \mathbf{a_{n+1\ n+1}} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Pseudo difference operators 2

- To $\{d(s)|s \in \mathbb{Z}\}$ in R is associated $\text{diag}(d(s))$:

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{d(n-1)} & 0 & 0 & \ddots \\ \ddots & 0 & \mathbf{d(n)} & 0 & \ddots \\ \ddots & 0 & 0 & \mathbf{d(n+1)} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

- Diagonal matrices:

$$\mathcal{D}_1(R) = \{d = \text{diag}(d(s))|d(s) \in R \text{ for all } s \in \mathbb{Z}\}.$$

Pseudo difference operators 3

- Shift matrix Λ

$$\Lambda = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{0} & 1 & 0 & \ddots \\ \ddots & 0 & \mathbf{0} & 1 & \ddots \\ \ddots & 0 & 0 & \mathbf{0} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

- Action of the $\{\Lambda^m \mid m \in \mathbb{Z}\}$ on $\mathcal{D}_1(R)$:

$$\Lambda^m \mathrm{diag}(d(s))\Lambda^{-m} = \mathrm{diag}(d(s+m)).$$

- Each $A = (a_{ij}) \in M_{\mathbb{Z}}(R)$: decomposes uniquely

$$A = \sum_{i \in \mathbb{Z}} d_i \Lambda^i, d_i \in \mathcal{D}_1(R)$$

Pseudo difference operators 4

- Lower triangular matrices

$$LT(R) = \{L \mid L = \sum_{i \leq N} \ell_i \Lambda^i, \ell_i \in \mathcal{D}_1(R)\}$$

- Each $L = \sum_{i \leq N} \ell_i \Lambda^i, \ell_N \in \mathcal{D}_1(R)^*$, is invertible.
- Consider a $L_0 = \sum_{i \leq 1} \ell_i \Lambda^i, \ell_1 \in \mathcal{D}_1(R)^*$. Then:

$$L_0 = K_0 \Lambda K_0^{-1},$$

with $K_0 = \sum_{i \leq 0} k_i \Lambda^i, k_i \in \mathcal{D}_1(R), k_0 \in \mathcal{D}_1(R)^*$ and

$$LT(R) = \{P \mid P = \sum_{i \leq N} p_i L_0^i, p_i \in \mathcal{D}_1(R)\}$$

Pseudo difference operators 5

- Consider the invertible operator $\Delta := \Lambda - \mathrm{Id}$:

$$\Delta\left(\begin{pmatrix} \vdots \\ x_{n-1} \\ x_n \\ x_{n+1} \\ \vdots \end{pmatrix}\right) = \begin{pmatrix} \vdots \\ x_n - x_{n-1} \\ x_{n+1} - x_n \\ x_{n+2} - x_{n+1} \\ \vdots \end{pmatrix}$$

- For the difference operator Δ we have

$$\mathrm{Ps}\Delta = LT(R) = \{L \mid L = \sum_{i \leq N} \ell_i \Delta^i, \ell_i \in \mathcal{D}_1(R)\}$$

Elements of $\mathrm{Ps}\Delta$ also called: *pseudo difference operators*.

Compatible Lax equations in $\mathrm{Ps}\Delta$ 1

- Given R , set $\{\partial_i \mid i \geq 1\}$ of commuting derivations of R
- Example: $R = k[t_i \mid i \geq 1]$ or $R = k[[t_i \mid i \geq 1]]$ and

$$\partial_i := \partial_{t_i} := \frac{\partial}{\partial t_i}.$$

- Consider $L_0 = \sum_{i \leq 1} \ell_i \Lambda^i = K_0 \Lambda K_0^{-1}, \ell_1 \in \mathcal{D}_1(R)^*$, *with*

$$\partial_i(K_0) = 0, \text{ for all } i \geq 1.$$

- Define the Lie algebra

$$LT_{\geq 0}(L_0) = \{L \in \sum_{i \geq 0} l_i L_0^i, l_i \in K_0 \mathcal{D}_1(R) K_0^{-1}\}$$

and similarly $LT_{<0}(L_0)$, $LT_{>0}(L_0)$ and $LT_{\leq 0}(L_0)$.

Compatible Lax equations in $\mathrm{Ps}\Delta$ 2

- Sufficient to consider two decompositions in $\mathrm{Ps}\Delta$. First:

$$LT_{\geq 0}(\Lambda) \oplus LT_{<0}(\Lambda) = \mathrm{Ps}\Delta_{\geq 0} \oplus \mathrm{Ps}\Delta_{<0} = \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

- Group corresponding to $\mathfrak{g}_2 = \mathrm{Ps}\Delta_{<0}$:

$$U_- = \{\mathrm{Id} + B \mid B \in \mathrm{Ps}\Delta_{<0}\}$$

- Basic commuting directions : the $\{\Lambda^k \mid k \geq 1\}$
- Deformation of Λ:

$$\mathfrak{L} = \Lambda + \sum_{i=1}^{\infty} d_i \Lambda^{1-i}, d_i \in \mathfrak{D}_1(R).$$

- Examples: $\mathfrak{L} = U\Lambda U^{-1}$, with $U \in U_-$.

Compatible Lax equations in $\mathrm{Ps}\Delta$ 3

- Let $\mathcal{B}_r := (\mathcal{L}^r)_{\geq 0}, r \geq 1$.
- Search for deformations $\mathcal{L}$ that satisfy:

$$\partial_{k_1}(\mathcal{L}^{k_2}) = [\mathcal{B}_{k_1}, \mathcal{L}^{k_2}] = [\mathcal{L}^{k_2}, \mathcal{L}^{k_1}_{\leq 0}], k_1 \text{ and } k_2 \geq 1.$$

- Sufficient the Lax equations for $\mathcal{L}$

$$\partial_{k_1}(\mathcal{L}) = [\mathcal{B}_{k_1}, \mathcal{L}] = [\mathcal{L}, \mathcal{L}^{k_1}_{<0}], k_1 \geq 1,$$

  the **Lower Triangular Toda (LTT)-hierarchy**.
- Consequence: *zero curvature relations*

$$\partial_{k_1}(\mathcal{B}_{k_2}) - \partial_{k_1}(\mathcal{B}_{k_1}) - [\mathcal{B}_{k_1}, \mathcal{B}_{k_2}] = 0, k_1 \text{ and } k_2 \geq 1.$$

Compatible Lax equations in $\mathrm{Ps}\Delta$ 4

- Other relevant decomposition in $\mathrm{Ps}\Delta$:

$$LT_{>0}(\Lambda) \oplus LT_{\leq 0}(\Lambda) = \mathrm{Ps}\Delta_{>0} \oplus \mathrm{Ps}\Delta_{\leq 0} = \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

- Group corresponding to $\mathfrak{g}_2 = \mathrm{Ps}\Delta_{\leq 0}$:

$$P_- = \{d \,\mathrm{Id} + B \mid, d \in \mathcal{D}_1(R)^*, B \in \mathrm{Ps}\Delta_{<0}\}.$$

- Basic commuting directions : the $\{\Lambda^k \mid k \geq 1\}$.
- Deformation of Λ:

$$\mathcal{M} = d_0\Lambda + \sum_{i=1}^{\infty} d_i\Lambda^{1-i}, d_i \in \mathcal{D}_1(R) \text{ and } d_0 \in \mathcal{D}_1(R)^*.$$

- Examples: $\mathcal{M} = P\Lambda P^{-1}$, with $P \in P_-$.

Compatible Lax equations in PsΔ 5

- Consider the cut-off's $\mathcal{C}_r := (\mathcal{M}^r)_{>0}, r \geq 1$.
- Search for deformations $\mathcal{M}$ that satisfy:

$$\partial_{r_1}(\mathcal{M}^{r_2}) = [\mathcal{C}_{r_1}, \mathcal{M}^{r_2}] = [\mathcal{M}^{r_2}, \mathcal{M}^{r_1}_{\leq 0}], r_1 \text{ and } r_2 \geq 1.$$

- Sufficient Lax equations for $\mathcal{M}$ the

$$\partial_{r_1}(\mathcal{M}) = [\mathcal{C}_{r_1}, \mathcal{M}] = [\mathcal{M}, \mathcal{M}^{r_1}_{\leq 0}], r_1 \geq 1,$$

  the **Strict Lower Triangular Toda (SLTT)-hierarchy**.
- Consequence: *zero curvature relations*

$$\partial_{r_1}(\mathcal{C}_{r_2}) - \partial_{r_2}(\mathcal{C}_{r_1}) - [\mathcal{C}_{r_1}, \mathcal{C}_{r_2}] = 0, r_1 \text{ and } r_2 \geq 1.$$

Compatible Lax equations in $\mathrm{Ps}\Delta$ 6

- $\mathcal{L}$ solution of the LTT-hierarchy, $\mathcal{A}_k := -(\mathcal{L}^k)_{<0}, k \geq 1$.
- Zero curvature relations for the $\{\mathcal{A}_k \mid k \geq 1\}$:

$$\partial_{k_1}(\mathcal{A}_{k_2}) - \partial_{k_2}(\mathcal{A}_{k_1}) - [\mathcal{A}_{k_1}, \mathcal{A}_{k_2}] = 0, k_1 \text{ and } k_2 \geq 1.$$

- $\mathcal{M}$ solution of the SLTT-hierarchy, $\mathcal{D}_r := -(\mathcal{M}^r)_{\leq 0}, r \geq 1$.
- Zero curvature for the $\{\mathcal{D}_r \mid r \geq 1\}$:

$$\partial_{r_1}(\mathcal{D}_{r_2}) - \partial_{r_2}(\mathcal{D}_{r_1}) - [\mathcal{D}_{r_1}, \mathcal{D}_{r_2}] = 0, r_1 \text{ and } r_2 \geq 1.$$

Related Cauchy problems 1

- The setting $(R, \{\partial_i\})$ is called *exponentially complete*, if for each $r \in R$ the element
$$e^r = \sum_{j=0}^{\infty} \frac{r^j}{j!}$$
is a well-defined element of R^*, satisfying $\partial_i(e^r) = \partial_i(r)e^r$.
- The setting $(R, \{\partial_i\})$ is said to be *compatibility complete*, if for each collection $\{g(i) \in R, i = 1, 2, \cdots\}$ that satisfies the compatibility conditions
$$\partial_{i_1}(g(i_2)) = \partial_{i_2}(g(i_1)), \text{ for all } i_1 \text{ and } i_2 \geq 1, \tag{2}$$
there exists a $\kappa \in R$ satisfying
$$\partial_i(\kappa) = g(i) \text{ for all } i \geq 1.$$
- Example: the setting $(k[[t_i \mid i \geq 1]], \{\partial_i := \frac{\partial}{\partial t_i})$.

Related Cauchy problems 2

- Assume we have a set of pseudo difference operators of strict negative order
$$\{A_i \mid i \geq 1, A_i = \sum_{m>0} a_m(i)\Lambda^{-m}\}.$$
For example, $\{A_i = -(\mathcal{L}^i)_{<0}\}$ with $\mathcal{L}$ a potential solution of the LTT-hierarchy..
- Consider for a $\mathcal{K} = 1 + \sum_{j<0} k_j \Lambda^j$, $k_j \in \mathcal{D}_1(R)$, the system
$$\partial_i(\mathcal{K}) = A_i \mathcal{K}, i \geq 1, \quad (3)$$

Theorem

If the standard setting $(R, \{\partial_i\})$ is compatibility complete, then there is a solution $\mathcal{K}$ in PsΔ of the system (3) if and only if the $\{A_i\}$ satisfy the zero curvature relations.

Related Cauchy problems 3

- Assume we have a set of pseudo difference operators of order zero or less
$$\{D_r \mid r \geqslant 1, D_r = \sum_{m\geqslant 0} d_m(i)\Lambda^{-m}\}.$$
For example, $\{D_r = -(\mathcal{M}^r)_{\leqslant 0}\}$ with $\mathcal{M}$ a potential solution of the SLTT-hierarchy.
- Consider for a $\mathcal{K} = \sum_{j\leqslant 0} k_j\Lambda^j$, $k_i \in \mathcal{D}_1(R)$, $k_0 \in \mathcal{D}_1(R)^*$, the system
$$\partial_r(\mathcal{K}) = D_r\mathcal{K}, r \geqslant 1, \tag{4}$$

Theorem

Let the setting $(R, \{\partial_i\})$ be compatibility complete and exponentially complete. Then there is a solution $\mathcal{K}$ in PsΔ of the system (4) if and only if the $\{D_r\}$ satisfy the zero curvature relations.

Geometric construction of solutions 1

- Hilbert space

$$H = \{\sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 < \infty\},$$

- Subspaces $H_i = \{\sum_{n\leq i} a_n z^n \in H\}, i \in \mathbb{Z}$.
- Orthonormal basis: $\{e_i := z^i \mid i \in \mathbb{Z}\}$
- $b \in B(H) \Rightarrow \mathbb{Z} \times \mathbb{Z}$-matrix $[b] = (b_{ij})$ w.r.t. the $\{e_i\}$.
- Example: Λ is the matrix corresponding to $M_{\frac{1}{z}} : H \to H$,

$$M_{\frac{1}{z}}(\sum_{n\in\mathbb{Z}} a_n z^n) = \sum_{n\in\mathbb{Z}} a_n z^{n-1}.$$

Geometric construction of solutions 2

- Each $b \in B(H)$ decomposes as $b = u_-(b) + p_+(b)$, with

$$[u_-(b)] = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{0} & 0 & 0 & \ddots \\ \ddots & b_{n\ n-1} & \mathbf{0} & 0 & \ddots \\ \ddots & b_{n+1\ n-1} & b_{n+1\ n} & \mathbf{0} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

$$[p_+(b)] = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{b_{n-1\ n-1}} & b_{n-1\ n} & b_{n-1\ n+1} & \ddots \\ \ddots & 0 & \mathbf{b_{n\ n}} & b_{n\ n+1} & \ddots \\ \ddots & 0 & 0 & \mathbf{b_{n+1\ n+1}} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Geometric construction of solutions 3

- Consider the group

$$G(0) = \{g \in \mathrm{GL}(H) \mid u_-(g) \text{ and } u_-(g^{-1}) \text{ Hilbert-Schmidt } \}.$$

- For $n \in \mathbb{Z}$, put $G(n) = (M_{\frac{1}{z}})^n G(0)$.
- $G = \cup_{n \in \mathbb{Z}} G(n)$ group, each $G(n)$ connected component of G
- Commuting flows in G:

$$\Gamma = \{\gamma(t) = \exp(\sum_{i=1}^{\infty} t_i (M_{\frac{1}{z}})^i)\}$$

Geometric construction of solutions 4

- Big cell in $G(0)$: $\Omega = U_- P_+ = P_- U_+$, where:
  - $U_- := \{g = \mathrm{Id} + u_-(g) \mid g \in G(0)\}$,
  - $P_+ := \{g = p_+(g) \mid g \in G(0)\}$,
  - $U_+ := \{g \in P_+ \mid g_{ii} = 1, \text{ for all } i \in \mathbb{Z}\}$,
  - $P_- := \{g \in G(0) \mid g_{ij} = 0 \text{ for all } i < j\}$
- Big cell in $G(n)$: $(M_{\frac{1}{z}})^n \Omega = (M_{\frac{1}{z}})^n U_- P_+ = (M_{\frac{1}{z}})^n P_- U_+$
- For $g \in G(n)$, choose the algebra of coefficients

$$R_g := C^\infty(\{\gamma \in \Gamma \mid \gamma(t) g \gamma(t)^{-1} \in (M_{\frac{1}{z}})^n \Omega\}),$$

with the derivations $\partial_i = \frac{\partial}{\partial t_i}, i \geq 1$.

Geometric construction of solutions 5

Theorem

There holds:

(a) *Let $g \in G$. For each coset $gP_+ \in G/P_+$ there is a $\mathcal{L}_{gP_+}$ in PsΔ that is a solution of the LTT-hierarchy.*

(b) *Let $g \in G$. For each coset $gU_+ \in G/U_+$ there is a $\mathcal{M}_{gU_+}$ in PsΔ that is a solution of the SLTT-hierarchy.*

- For $i \in \mathbb{Z}$, define the subspace

$$H_i := \{\sum_{n \leq i} a_n z^n \in H\}.$$

- The $\{H_i\}$ form the basic flag

$$\cdots H_{i-1} \subset H_i \subset H_{i+1} \cdots ,$$

corresponding to Id P_+.

Geometric construction of solutions 6

- To gP_+ corresponds the flag $\mathcal{F}_{gP_+} = \{W_i = gH_i\}$:

$$\cdots gH_{i-1} \subset gH_i \subset gH_{i+1} \cdots$$

- To gU_+ corresponds the flag $\mathcal{F}_{gP_+} = \{W_i = gH_i\}$ and the basis $\{f_i\}$,

$$f_i \neq 0, f_i \in W_i/W_{i-1}.$$

Darboux transformations 1

- R ring of functions, $\partial = \frac{\partial}{\partial t_1}$ derivation of R
- $v \in R$, Schrödinger operator $\mathcal{L}_2 := \partial^2 + v$,
- $P_3 = \partial^3 + \frac{3}{2} v\partial + \frac{3}{4} = (\mathcal{L}_2)^{\frac{3}{2}}_{\geq 0}$
- The Lax equation

$$\frac{\partial}{\partial t_3}(\mathcal{L}_2) = [P_3, \mathcal{L}_2]$$

is equivalent with the KdV-equation:

$$\frac{\partial}{\partial t_3}(v) = \frac{1}{4}\frac{\partial^3}{\partial t_1^3}(v) + \frac{3}{2} v \frac{\partial}{\partial t_1}(v)$$

for v.

Darboux transformations 2

- Consider nonzero ϕ in kernel of $\mathfrak{L}_2$

$$\mathfrak{L}_2(\phi) = \partial^2(\phi) + v\phi = 0 \text{ and } \phi^{-1} \text{ exists .}$$

- $\mathfrak{L}_2$ decomposes as

$$\mathfrak{L}_2 = (\partial + \frac{\partial(\phi)}{\phi})(\partial - \frac{\partial(\phi)}{\phi})$$

- Darboux transformation of $\mathfrak{L}_2$

$$\begin{aligned}\tilde{\mathfrak{L}}_2 =&(\partial - \frac{\partial(\phi)}{\phi})(\partial + \frac{\partial(\phi)}{\phi}) = \partial^2 + \tilde{u} \\ =&(\partial - \frac{\partial(\phi)}{\phi})\mathfrak{L}_2(\partial - \frac{\partial(\phi)}{\phi})^{-1}\end{aligned}$$

Darboux transformations 3

- Compatibility with KdV: if $q := \partial(\phi) \cdot \phi^{-1}$ satisfies

$$\frac{\partial q}{\partial t} = \partial^3(q) - 6q\partial(q),$$

  and v satisfies the KdV-equation, then also $\tilde{v}$ satisfies the KdV-equation.
- Generalizing to KP, this led to the question:

> Determine in a class of solutions L of the KP hierarchy differential operators $P \in R[\partial]$ such that PLP^{-1} is again a solution of the KP hierarchy in this class.

Darboux transformations 3

- Similar question in PsΔ:

  Determine in a class of solutions $\mathcal{L}$ resp. $\mathcal{M}$ of the LTT hierarchy resp. SLLT hierarchy difference operators $\mathcal{P}$, resp. $\mathcal{Q} \in \mathrm{Ps}\Delta_{\geq 0}$ such that $\mathcal{P}\mathcal{L}\mathcal{P}^{-1}$ resp. $\mathcal{Q}\mathcal{M}\mathcal{Q}^{-1}$ is again a solution of the LTT hierarchy resp. SLTT hierarchy.

- If the order of $\mathcal{P}$ in Λ is n, then the transformations

$$\mathcal{L} \to \mathcal{P}\mathcal{L}\mathcal{P}^{-1} \text{ and } \mathcal{M} \to \mathcal{Q}\mathcal{M}\mathcal{Q}^{-1}$$

  are called *Darboux transformations of order n*.

Darboux transformations 4

Theorem

The Darboux transformations of order n:

(a) *Let $g_1, g_2 \in G$. The solutions $\mathcal{L}_{g_1 P_+}$ and $\mathcal{L}_{g_2 P_+}$ of the LTT hierarchy are connected by a Darboux transformations of order n if and only if the corresponding flags $\mathfrak{F}_{g_1 P_+} = \{V_i\}$ and $\mathfrak{F}_{g_2 P_+} = \{W_i\}$ satisfy for all $i \in \mathbb{Z}$:*

$$V_i \subset W_i, \ dim(W_i/V_i) = n.$$

(b) *Let $g_1, g_2 \in G$. The solutions $\mathcal{M}_{g_1 U_+}$ and $\mathcal{M}_{g_2 U_+}$ of the SLTT hierarchy are connected by a Darboux transformations of order n if and only if the corresponding flags $\mathfrak{F}_{g_1 P_+} = \{V_i\}$ and $\mathfrak{F}_{g_2 P_+} = \{W_i\}$ satisfy for all $i \in \mathbb{Z}$:*

$$V_i \subset W_i, \ dim(W_i/V_i) = n.$$

THANK YOU FOR YOUR ATTENTION