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Dispersionless Integrability in dimension five    Joint work with Omid Makhmali

$$F\in C^\infty(J^\ell M)\overset{loc}{\rightsquigarrow}\mathcal{E}=\{F=0\}\subset J^\ell M\overset{prol}{\rightsquigarrow}\mathcal{E}_\infty=\{D_\alpha F=0\}\subset J^\infty M$$

Symbol $$\sigma_F=\sum_{|\alpha|=\ell}\frac{\partial F}{\partial u_\alpha}\partial_\alpha\in\Gamma(\pi_\infty^* S^\ell TM_u)$$

Here $M\simeq M_u:=j^\infty_u(M)\subset\mathcal{E}_\infty$
for $u\in Sol(\mathcal{E})$

For 2nd ord operator:

$$\sigma_F=\sum_{i\le j}\frac{\partial F}{\partial u_{ij}}\partial_i\partial_j=\sum_{i,j}\sigma^{ij}(u)\partial_i\partial_j$$

$$Char(\mathcal{E})=\{\sigma_F(p)=\sum_{i,j}\sigma^{ij}(u)p_ip_j=0\}\subset\mathbb{P}T^*M_u$$

quadratic characteristic variety

If the symbol (or Char) is nondegenerate i.e. $\det[\sigma^{ij}(u)]\neq0$

then $g_F=\sum_{i,j}g_{ij}(u)dx^idx^j$, $(g_{ij}(u))=(\sigma^{ij}(u))^{-1}$, is a metric.

The conformal structure $C_\mathcal{E}=[g_F]$ on $M_u$ depends only on $\mathcal{E}$.
[E.Ferapontov–BK]

**Example: dKP** $F = u_{tx} - (uu_x)_x - u_{yy} \rightarrow \sigma_F = \partial_t\partial_x - u\partial_x^2 - \partial_y^2$

$$\left(\sigma^{ij}(u)\right) = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & -u & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad (g_{ij}) = \begin{pmatrix} 4u & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\mathrm{Char} = \{p_t p_x - u p_x^2 - p_y^2 = 0\}$

$g = 4u\,dt^2 + 4dt\,dx - dy^2$

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**Definition: dlp** Dispersionless Lax pair of ord $= k$ is a $\mathbb{P}^1$-fiber Bundle $\hat\pi : \hat M_u \longrightarrow M_u$ and rank 2 distribution $\hat\Pi \subset T\hat M_u$ such that

- $\forall\, \hat x \in \hat M_u$ $\quad \hat\Pi(\hat x)$ depends only on $j^k_{x} u$
- $\hat\Pi = \langle \hat X = X + m\partial_\lambda,\ \hat Y = Y + n\partial_\lambda \rangle \pitchfork \hat\pi_*^{-1}(\cdot) = \langle \partial_\lambda \rangle$
- Frob. $\int$ : $[\hat\Pi, \hat\Pi] = \hat\Pi \mod \mathcal{E}$

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Twistor fibration

$$\begin{tikzcd} & \hat M_u^{n+1} \arrow[dl, "\hat\pi^2"'] \arrow[dr, "\mathbb{P}^1"] & \\ T_u^{n-1} & & M_u^n \end{tikzcd}$$

correspondence space

twistor space

solution manifold

2 plane congruence $\Pi(\hat x) = d\hat\pi_{\hat x}(\hat\Pi) \subset T_x M$, $\hat x = (x, \lambda)$, is parametrized by the spectral parameter $\lambda$

**Theorem (D. Calderbank, BK)** dLp $\hat{\Pi}$ is characteristic i.e. $\forall u \in Sol(\mathcal{E})$ $\forall \hat{x} \in \hat{M}_u$ $\forall p \in \mathrm{Ann}\,\Pi(\hat{x}) \subset T^*_x M_u$ : $\sigma_F(p) = 0 \Leftrightarrow [p] \in \mathrm{Char}(\mathcal{E})$

This means that $\Pi(\hat{x})$ is coisotropic 2-plane for $C_{\mathcal{E}}$.
Such planes exist only for $2 \leq n \leq 4$ and in $T_x M_u$ they

$\boxed{n=2}$ everything $\boxed{n=3}$ rational conic $\mathbb{P}^1$ $\boxed{n=4}$ two disj. rat. curves $\mathbb{P}^1_\alpha \sqcup \mathbb{P}^1_\beta$

**Corollary** For $n>4$ PDEs $\mathcal{E} = \{F=0\}$ with nondegenerate quadratic $\mathrm{Char}(\mathcal{E})$ are not integrable via dLp.

**Ex:** Let $F_0$ be an integrable ndg diff op in 3D or 4D. Extension $F[u] = F_0[u] + \Delta u$ is not integrable.
For instance, this is the case dKP $\rightsquigarrow$ $KZ_{n \geq 4}$.

# IBG in 3D

Einstein-Weyl structure is a triple $([g], \omega, D)$

conf. str. — 1-form (Weyl poten) — sym conn

$$Dg = \omega \otimes g, \quad Ric^{sym}_D = \Lambda g$$

for some $\Lambda \in C^\infty(M)$

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Ex: EW from dKP $\quad u_{tx} = (uu_x)_x + u_{yy} \qquad \hat{M}_u \simeq \mathbb{R}^4(x,y,t,\lambda)$

$$g = 4dxdt - dy^2 + 4u\,dt^2, \quad \omega = -4u_x dt, \quad \hat{\Pi} = \langle \hat{X}, \hat{Y} \rangle$$

$$\hat{X} = \partial_y - \lambda\partial_x + u_x\partial_\lambda, \quad \hat{Y} = \partial_t - (\lambda^2 + u)\partial_x + (u_x\lambda + u_y)\partial_\lambda$$

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# IBG in 4D

Weyl tensor of neutral sign conf. str $[g]$ splits

$W = W_+ + W_-$ where $*W_\pm = \pm W_\pm$. $[g]$ is SD/ASD if $W_+ = 0 / W_- = 0$.

Ex: self-dual gravity The second Plebański eqn

$u_{xz} + u_{yt} + u_{xx}u_{yy} - u_{xy}^2 = 0$ has SD str on sol $M_u \simeq \mathbb{R}^4(x,y,z,t)$

$$g = dxdz + dydt - u_{yy}dz^2 + 2u_{xy}dzdt - u_{xx}dt^2$$

$$\hat{X} = \partial_t + u_{xx}\partial_y - (u_{xy} - \lambda)\partial_x \qquad \hat{Y} = \partial_z - (u_{xy} + \lambda)\partial_y + u_{yy}\partial_x$$

For integrable PDEs with quadratic Char($\mathcal{E}$) the maximal rank of $\sigma_F \in S^2 TM_u$ is 4 i.e. the distribution

$$\Delta_{\mathcal{E}} = \langle \sigma_F, T^*M_u \rangle \subset TM_u$$

(obtained by contraction) has (max) rank 4.

In general $\Delta_{\mathcal{E}}$ is nonholonomic distribution for generic $u \in Sol(\mathcal{E})$. The symm bivector $\sigma_F \in S^2 \Delta_{\mathcal{E}}$ is nondegenerate. It inverse $g_F = \sigma_F^{-1}$ defines canonical subconformal structure $c_{\mathcal{E}} = [g_F]$ on $\Delta_{\mathcal{E}}$

Nonholonomic distributions can have various weak derived flags and symbols but most of them are incompatible with the requirements of IBG.

In what follows we restrict to contact 5D $(M_u, \Delta_{\mathcal{E}})$.

**Data:** $(M,\Delta)$ contact 5-manf, $g\in\Gamma(S^2\Delta^*)$ conf. metric of neutral sign, $\Omega\in\Gamma(\Lambda^2\Delta^*)$ conf. sympl. form, + orientation

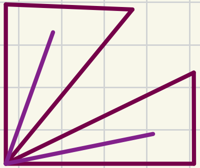
$J=g^{-1}\Omega$ endomorphism of $\Delta$ normalized by $\|J\|=1$.

Subconf. str $(M,\Delta,[g])$ is compatible if $\alpha$ (self-dual) family consist of $g$-null $\Omega$-Lagrangian planes $(\mathbb{P}^1_\alpha)$.

(Then $\beta$-family contains 2 v 0 such planes.)

$$g(Jv,w)+g(v,Jw)=0 \qquad \forall v,w\in\Delta$$

$$J^2=\delta\cdot\mathbb{1}\ ,\quad \delta=\pm1\ ,\quad \mathrm{Tr}(J)=0$$



$\underline{\delta=+1|}$ $g$-null $\Omega$-Lagrangian planes are either eigenspaces $L_\pm=E_J(\pm1)$ or family $\Pi=\langle\ell_-,\ell_+\rangle$, $\ell_\pm\in L_\pm$, $\ell_-\perp\ell_+$

$\underline{\delta=-1|}$ $g$-null $\Omega$-Lagrangian planes form 1-parameter family $\Pi$ of $J$-inv null planes

$SL(2,\mathbb{R})$ reparametr. $\lambda\in\mathbb{P}^1$

Compatible subconformal geom with $\delta=+1$ is a Lagrangian contact geometry $\Delta = L_-\oplus L_+$ and is a parabolic geom of type $(SL(4,\mathbb{R}), P_{13})$

$$\begin{bmatrix} 0 & +1 & +1 & +2 \\ -1 & 0 & 0 & +1 \\ -1 & 0 & 0 & +1 \\ -2 & -1 & -1 & 0 \end{bmatrix}$$

This Cartan geometry has harmonic curv:

$$\kappa_H = \tau_- + \tau_+ + W$$

torsions curvature

$$\tau_\pm \in \Gamma\left(\Lambda^2 L^*_{\mp} \otimes L_\pm\right) \qquad W \in \Gamma\left(\left(S^4 L^*_- \otimes \left(\Lambda^2 L^*_-\right)^{-2} + S^4 L^*_+ \otimes \left(\Lambda^2 L^*_+\right)^{-2}\right) \otimes \Delta^{1}\right)$$

2 components each  5 components

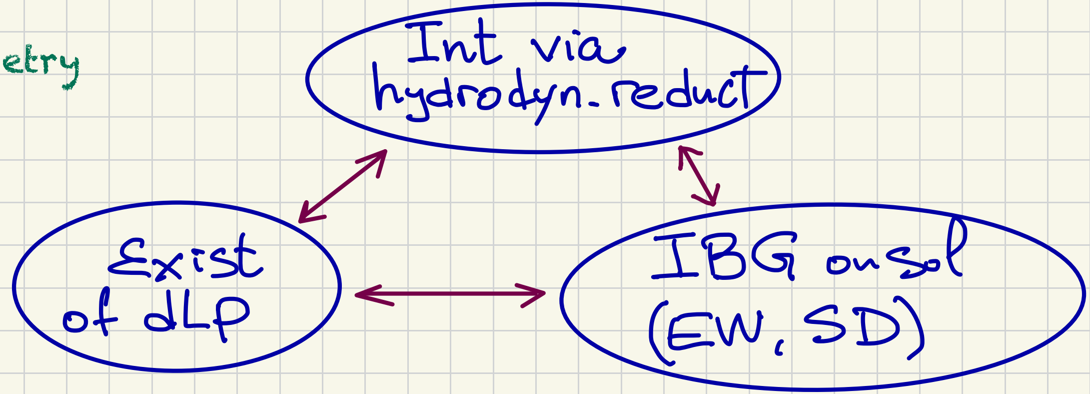
Similarly $\delta=-1$ yields almost CR hypers type geom $\Delta = L\otimes\mathbb{C}$ i.e. a parabolic geom of type $(SU(2,2), P_{13})$

$$\kappa_H = \tau_{\mathbb{C}} + W$$

# Integrability via Geometry

## 3D + 4D

E. Ferapontov + BK
D. Calderbank + BK



Int via hydrodyn. reduct

Exist of dLP

IBG on sol (EW, SD)

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## 5D

BK + O. Makhmali

The integrability of $\mathcal{E}$ via dLp is equivalent to the zero-curvature $\mathbb{W}=0 \quad \forall\, u \in Sol(\mathcal{E})$.

Note that $\mathbb{W}$ (similar to $\mathbb{W}_+$ and $\mathbb{W}_-$ in 4D) is quartic

$$\mathbb{W} = \mathbb{W}_0 + \mathbb{W}_1 \lambda + \mathbb{W}_2 \lambda^2 + \mathbb{W}_3 \lambda^3 + \mathbb{W}_4 \lambda^4$$

Example: 5D version of the Plebansky eqn

$$F = u_{15} + u_{13}u_{24} - u_{14}u_{23}$$

$$\begin{aligned} \ell_F &= \partial_1\partial_5 + u_{13}\partial_2\partial_4 + u_{24}\partial_1\partial_3 - u_{14}\partial_2\partial_3 - u_{23}\partial_1\partial_4 \\ \| \\ \sigma_F &= V_1 \cdot V_3 + V_2 \cdot V_4 \end{aligned}$$

for $V_1 = \partial_1$ $\quad V_3 = \partial_5 - u_{23}\partial_4 + u_{24}\partial_3$

$V_2 = \partial_2$ $\quad V_4 = u_{13}\partial_4 - u_{14}\partial_3$

$$\Delta = \langle V_1, V_2, V_3, V_4 \rangle = \mathrm{Ann}(\omega^0)$$

$$\omega^0 = dx^3 + \frac{u_{14}}{u_{13}}dx^4 + \frac{u_{15}}{u_{13}}dx^5$$

$$g = \sigma_F^{-1} = \omega^1\omega^3 + \omega^2\omega^4$$

$\omega^1 = dx^1$ $\quad \omega^3 = dx^5$

$\omega^2 = dx^2$ $\quad \omega^4 = \frac{1}{u_{13}}dx^4 + \frac{u_{23}}{u_{13}}dx^5$

$\alpha$-planes

$$\{\omega^1 = -\lambda\omega^4, \omega^2 = \lambda\omega^3\} = \langle V_4 - \lambda V_1, V_3 + \lambda V_2 \rangle$$

$$\hat{\Pi} = \langle \partial_5 - u_{23}\partial_4 + u_{24}\partial_3 + \lambda\partial_2,\ u_{13}\partial_4 - u_{14}\partial_3 - \lambda\partial_1 \rangle$$

# Master equation for integrable systems in 5D

General zero-curvature subconformal structures on contact distributions in 5D are locally:

$$\omega^0 = \underline{dr - p}\,dx - q\,dy$$

$$g = (dx + (u+v)dy)\cdot(dq - (u-v)dp + w\,dx + (z - w(u-v))dy)$$
$$+ (dx + (u-v)dy)\cdot(dq - (u+v)dp + w\,dx + (z - w(u+v))dy)$$

$x, y, p, q, r$ : indep. var

$u, v, w, z$ : dep. var

$$\boxed{\Rightarrow W_4 = 0}$$

These must satisfy 4 second ord PDEs

$$W_0 = W_1 = W_2 = W_3 = 0$$

3D| 4 funct 2 arg     4D| 6 funct 3 arg     5D| 8 funct 4 arg