Classification of Homogeneous Third Order Differential-Geometric Poisson Brackets

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First Order Differential-Geometric Poisson Brackets

- First order differential-geometric Poisson brackets

$$\{u^i(x), u^j(x')\} = [g^{ij}(\mathbf{u}(x))\partial_x + b^{ij}_k(\mathbf{u}(x))u^k_x]\delta(x-x')$$

satisfy the skew-symmetry and Jacobi identity iff $g^{ij}(\mathbf{u})$ is a symmetric nondegenerate tensor and $b^{ij}_k(\mathbf{u}) = -g^{is}\Gamma^j_{sk}$, where Γ^j_{sk} is a Levi-Civita connection, while the metric $g_{ij}(\mathbf{u})$ is flat.

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- Corresponding Hamiltonian systems are

$$u^i_t = [g^{ij}\partial_x - g^{is}\Gamma^j_{sk}u^k_x]\frac{\delta \mathbf{H}}{\delta u^j},$$

where the Hamiltonian functional $\mathbf{H} = \int h(\mathbf{u}, \mathbf{u}_x, \mathbf{u}_{xx}, \mathbf{u}_{xxx}, ...)dx$.

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- In a particular case $\mathbf{H} = \int h(\mathbf{u})dx$, we have N component hydrodynamic type system

$$u_t^i = (\nabla^i\nabla_j h)u_x^j.$$

Third Differential-Geometric Poisson Brackets

$$\{u^i(x), u^j(x')\} = [g^{ij}(\mathbf{u}(x))\partial_x^3 + b_k^{ij}(\mathbf{u}(x))u_x^k\partial_x^2$$
$$+(c_k^{ij}(\mathbf{u}(x))u_{xx}^k + c_{km}^{ij}(\mathbf{u}(x))u_x^k u_x^m)\partial_x$$
$$+d_k^{ij}(\mathbf{u}(x))u_{xxx}^k + d_{km}^{ij}(\mathbf{u}(x))u_x^k u_{xx}^m + d_{kmn}^{ij}(\mathbf{u}(x))u_x^k u_x^m u_x^n]\delta(x - x').$$

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where $c_{nkm} = \frac{1}{3}(g_{mn,k} - g_{kn,m})$ and $c_{ijk} = g_{iq}g_{jp}b_k^{pq}$.

Metric. Nonlinear System

$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0,$$

$$g_{mn,kl} = \frac{1}{9} g^{pq} \big(g_{qk,n} g_{pl,m} - g_{qk,n} g_{pm,l} + g_{qk,m} g_{pl,n} - g_{qk,m} g_{pn,l} + g_{qn,k} g_{pm,l} - g_{qn,k} g_{pl,m} + g_{qm,k} g_{pn,l} - g_{qm,k} g_{pl,n} \big).$$

- This system has a solution in the form

$$g_{ik} = g^{(0)}_{ik} + g^{(1)}_{ikm} a^m + g^{(2)}_{ikmn} a^m a^n.$$

- The nonlinear part

$$g_{mn,kl} = \frac{1}{9} g^{pq} \big(g_{qk,n} g_{pl,m} - g_{qk,n} g_{pm,l} + g_{qk,m} g_{pl,n} - g_{qk,m} g_{pn,l} + g_{qn,k} g_{pm,l} - g_{qn,k} g_{pl,m} + g_{qm,k} g_{pn,l} - g_{qm,k} g_{pl,n} \big)$$

can be parameterized in the form

$$g_{ik} = \phi^{\beta\gamma} \psi_{i\beta} \psi_{k\gamma},$$

where $\psi_{i\beta} = \psi_{i\beta m} a^m + \xi_{i\beta}$ and $\psi_{i\beta k} = -\psi_{k\beta i}$.

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- The linear part

$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0$$

can be solved by virtue of the Monge parameterization:

$$ds^2 = g_{ik}(\mathbf{a})da^i da^k = \vec{d}^T \hat{Q} \vec{d},$$

where $\vec{d} = (da^1, da^2, ..., da^N, a^1 da^2 - a^2 da^1, a^2 da^3 - a^3 da^2, ...)$ and $\hat{Q}$ is a constant nondegenerate symmetric matrix.

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- Passive Form.

$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0,$$

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Particular Case

Theorem: *In the particular case* $c_{ijk} = 0$ *(three distinct indices), a general solution of the above nonlinear system is parameterized by a sole polynomial function* G *of degree* 4 *such that*

$$g_{kk} = -2G_{,kk}, \quad g_{km} = G_{,km}, \quad c_{kkm} = -c_{kmk} = G_{,kkm}, \quad k \neq m,$$

while all other connection coefficients $c_{ijk} = 0$.

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This leads to

$$g_{mm} = -\sum_{p \neq m} R_{mp}(a^p)^2 - 2\sum_{p \neq m} H_{mp}a^p + D_m,$$
$$g_{km} = R_{km}a^k a^m + H_{km}a^k + H_{mk}a^m + F_{km}, \; k \neq m,$$

where $R_{km} = R_{mk}$, $F_{km} = F_{mk}$, D_k are constants, and

$$c_{mmk} = -c_{mkm} = R_{km}a^k + H_{mk}, \; k \neq m.$$

Two Component Case

Theorem: only two metrics in 2-component case:

$$g_{ik}^{(1)} = \begin{pmatrix} 1-(a^2)^2 & 1+a^1a^2 \\ 1+a^1a^2 & 1-(a^1)^2 \end{pmatrix}, \quad g_{ik}^{(2)} = \begin{pmatrix} -2a^2 & a^1 \\ a^1 & 0 \end{pmatrix}.$$

Three Component Case

Using the ansatz $c_{ijk} = 0$ we are able to reduce the nonlinear system to:

$$R_{12}F_{12} = H_{12}H_{21}, \quad F_{13}R_{13} = H_{13}H_{31}, \quad F_{23}R_{23} = H_{23}H_{32},$$
$$D_1 R_{12}R_{13} + H_{12}^2 R_{13} + H_{13}^2 R_{12} = 0,$$
$$D_2 R_{12}R_{23} + H_{21}^2 R_{23} + H_{23}^2 R_{12} = 0,$$
$$D_3 R_{13}R_{23} + H_{31}^2 R_{23} + H_{32}^2 R_{13} = 0.$$

In the generic case $R_{ij} \neq 0$ we obtain the three-parameter family of metrics

$$g_{ik} = \begin{pmatrix} -(a^2+\beta_2)^2 - (a^3)^2 & a^1(a^2+\beta_2) & a^3(a^1+\beta_1) \\ a^1(a^2+\beta_2) & -(a^1)^2 - (a^3+\beta_3)^2 & a^2(a^3+\beta_3) \\ a^3(a^1+\beta_1) & a^2(a^3+\beta_3) & -(a^1+\beta_1)^2 - (a^2)^2 \end{pmatrix}$$

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The particular cases

1. $R_{12} = 0,\ R_{13} \neq 0,\ R_{23} \neq 0,$
2. $R_{12} = 0,\ R_{13} = 0,\ R_{23} \neq 0,$
3. $R_{12} = 0,\ R_{13} = 0,\ R_{23} = 0,$

were solved separately, obtaining a complete classification of 53 metrics.

WDVV Associativity Equations

- In three component case, WDVV associativity equations reduce to the single equation

$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt},$$

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$$f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt},$$
which can be written as the hydrodynamic type system
$$a_t = b_x, \quad b_t = c_x, \quad c_t = (b^2 - ac)_x,$$
where $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$.

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$$a_t = b_x, \quad b_t = c_x, \quad c_t = (b^2 - ac)_x,$$
where $a = f_{xxx}$, $b = f_{xxt}$, $c = f_{xtt}$.
- This system is bi-Hamiltonian, i.e.
$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}_t = J_0 \delta H_1 = J_1 \delta H_0,$$
where
$$J_0 = \begin{pmatrix} -\frac{3}{2}D & \frac{1}{2}Da & Db \\ \frac{1}{2}aD & \frac{1}{2}(bD + Db) & \frac{3}{2}cD + c_x \\ bD & \frac{3}{2}Dc - c_x & (b^2 - ac)D + D(b^2 - ac) \end{pmatrix}$$

WDVV Associativity Equations

- and

$$J_1 = \begin{pmatrix} 0 & 0 & D^3 \\ 0 & D^3 & -D^2 aD \\ D^3 & -DaD^2 & D^2 bD + DbD^2 + DaDaD \end{pmatrix}.$$

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Here $H_1 = \int c dx$, $H_0 = -\frac{1}{2} a(D^{-1}b)^2 - (D^{-1}b)(D^{-1}c)$.

Lorentzian Metric

The third order Hamiltonian structure is associated with the action

$$S = \frac{1}{2}\int (f_{xt}^2 f_{xxx} + f_{xt} f_{tt}) dxdt.$$

Riemann Curvature Tensor

The Riemann tensor of curvature has the form

$$R_{jikl} = g_{js} R^s_{ikl} = g_{js}[\partial_k \Gamma^s_{il} - \partial_l \Gamma^s_{ik} + \Gamma^s_{kn}\Gamma^n_{il} - \Gamma^s_{ln}\Gamma^n_{ik}],$$

where

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m}).$$

However, taking into account

$$g_{mk,n} + g_{kn,m} + g_{mn,k} = 0,$$

we obtain

$$\Gamma^i_{jk} = -g^{im} g_{jk,m}.$$

Then

$$R_{jikl} = g_{ik,jl} - g_{il,jk} + g^{sm}(g_{ik,m}g_{jl,s} - g_{il,m}g_{jk,s}).$$

References

eamericonarticle *A.V. Balandin, G.V. Potemin,*
On non-degenerate differential-geometric Poisson brackets of third order, Russian Mathematical Surveys **56** No. 5 (2001) 976-977.

eamericonarticle *P.W. Doyle,*
Differential geometric Poisson bivectors in one space variable, J. Math. Phys. **34** No. 4 (1993) 1314-1338.

eamericonarticle *B.A. Dubrovin and S.P. Novikov,*
Hamiltonian formalism of one-dimensional systems of hydrodynamic type and the Bogolyubov-Whitham averaging method, Soviet Math. Dokl., **27** (1983) 665–669.

eamericonarticle *E.V. Ferapontov,*
On integrability of 3×3 semi-Hamiltonian systems of hydrodynamic type which do not possess Riemann invariants, Physica D **63** (1993) 50-70.

References

eamericonarticle *E.V. Ferapontov,*
On the matrix Hopf equation and integrable Hamiltonian systems of hydrodynamic type which do not possess Riemann invariants, Phys. lett. A **179** (1993) 391-397.

eamericonarticle *E.V. Ferapontov,*
Several conjectures and results in the theory of integrable Hamiltonian systems of hydrodynamic type, which do not possess Riemann invariants, Teor. Math. Phys., **99** No. 2 (1994) 257-262.

eamericonarticle *E.V. Ferapontov,*
Dupin hypersurfaces and integrable Hamiltonian systems of hydrodynamic type, which do not possess Riemann invariants, Diff. Geom. Appl., **5** (1995) 121-152.

References

eamericonarticle *E.V. Ferapontov,*
Isoparametric hypersurfaces in spheres, integrable nondiagonalizable systems of hydrodynamic type, and *N*-wave systems, Diff. Geom. Appl., **5** (1995) 335-369.

eamericonarticle *E.V. Ferapontov, C.A.P. Galvao, O. Mokhov, Y. Nutku,*
Bi-Hamiltonian structure of equations of associativity in 2-d topological field theory, Comm. Math. Phys. **186** (1997) 649-669.

eamericonarticle *E.V. Ferapontov, O.I. Mokhov,*
Equations of associativity of two-dimensional topological field theory as integrable Hamiltonian nondiagonalisable systems of hydrodynamic type, Func. Anal. Appl. **30** No. 3 (1996) 195-203.

References

eamericonarticle *E.V. Ferapontov, R. Sharipov,*
On the first order conservation laws of hydrodynamic type systems, Teor. Math. Phys. **108** No. 1 (1996) 937-952.

eamericonarticle *O.I. Mokhov,*
Symplectic and Poisson structures on loop spaces of smooth manifolds, and integrable systems, Russian Math. Surveys **53** No. 3 (1998) 85–192.

eamericonarticle *S.P. Novikov,*
The geometry of conservative systems of hydrodynamic type. The method of averaging for field-theoretical systems, Russian Math. Surveys **40** No. 4 (1985) 85–98.

References

G.V. Potemin,
On Poisson brackets of differential-geometric type, Dokl. Akad. Nauk SSSR, **286** (1986) 39–42; Soviet Math. Dokl. **33** (1986) 30–33.

G.V. Potemin,
On third-order Poisson brackets of differential geometry, Russ. Math. Surv. **52** (1997) 617-618.

G.V. Potemin,
Some aspects of differential geometry and algebraic geometry in the theory of solitons. PhD Thesis, Moscow, Moscow State University (1991) 99 pages.