Generalized Toda and Volterra Systems

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Joint work with

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Classical Toda Lattice

Hamiltonian function

$$H(q_1, \ldots, q_N, p_1, \ldots, p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}.$$

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Lax pair $\dot{L} = [L_+, L]$ in Flaschka variables

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & & \cdots & a_{N-1} & b_N \end{pmatrix},$$

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- It follows that the functions $H_i = \frac{1}{i} \operatorname{Trace} L^i$ are constants of motion.
- Moreover, they are in involution with respect to a Poisson structure, associated to the above Lie algebra decomposition.

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The functions $H_i := \frac{1}{i} \operatorname{Trace} L^i$ are still in involution but they are not enough to ensure integrability.

Chopping

For $k = 0, \ldots, [\frac{(N-1)}{2}]$, denote by $(L - \lambda \,\mathrm{Id}_N)_k$ the result of removing the first k rows and the last k columns from $L - \lambda \,\mathrm{Id}_N$, and let

$$\det(L - \lambda \,\mathrm{Id}_N)_k = E_{0k}\lambda^{N-2k} + \cdots + E_{N-2k,k} \,.$$

Set

$$\frac{\det\,(L - \lambda \,\mathrm{Id}_N)_k}{E_{0k}} = \lambda^{N-2k} + I_{1k}\lambda^{N-2k-1} + \cdots + I_{N-2k,k} \,.$$

The functions I_{rk}, where $r = 1, \ldots, N-2k$ and $k = 0, \ldots, [\frac{N-1}{2}]$, are independent constants of motion, they are in involution and sufficient to account for the integrability of the full Toda lattice.

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- To these data one associates the Lax equation $\dot{L} = [L_+, L]$, where L and L_+ are defined as follows:

$$
\begin{aligned}
L &= \sum_{i=1}^{\ell} b_i H_{\alpha_i} + \sum_{i=1}^{\ell} a_i (X_{\alpha_i} + X_{-\alpha_i}), \\
L_+ &= \sum_{i=1}^{\ell} a_i (X_{\alpha_i} - X_{-\alpha_i}).
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- Ad-invariant functions on $\mathfrak{g}$ provide integrability

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The Lax equation takes the form

$$\dot{X} = [X_+, X],$$

where X_+ is the strictly lower triangular part of X, according to the Lie algebra decomposition strictly lower plus upper triangular.

Full-Kostant Toda lattice

The full-Kostant Toda lattice is obtained by replacing Π with Δ^+, in the sense that one fills the lower triangular part of X with additional variables. It leads on the affine space of all such matrices to the Lax equation

$$\dot{X} = [X_+, X],$$

where X_+ is again the projection to the strictly lower part of X.

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Consistency: the Lax matrix being symmetric, the bracket $[B_\Phi, L_\Phi]$ should give an element of the form

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In this case, we will say that Φ is adapted

Criterion for a set to be Adapted

The set Φ is adapted if and only if it satisfies the following property:

$$\forall \alpha, \beta \in \Phi, \alpha - \beta \text{ or } \beta - \alpha \in \Phi \cup \{0\}$$

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- $\Phi = \Pi$ corresponds to the classical Toda lattice
- $\Phi = \Delta^+$ corresponds to the full symmetric Toda.

Example B_2 Full Symmetric Toda

Roots

We consider a Lie algebra of type B_2. The set of positive roots $\Delta^+ = \{\alpha, \beta, \alpha+\beta, \beta+2\alpha\}$ which corresponds to the full symmetric Toda lattice with Lax matrix L.
This system is completely integrable with integrals $h_2 = \frac{1}{2}\mathrm{Tr}L^2$ which is the Hamiltonian, $h_4 = \frac{1}{2}\mathrm{Tr}L^4$ and a rational integral which is obtained by the method of chopping.

Lax Matrix

$$L = \begin{pmatrix} b_1 & a_1 & a_3 & a_4 & 0 \\ a_1 & b_2 & a_2 & 0 & -a_4 \\ a_3 & a_2 & 0 & -a_2 & -a_3 \\ a_4 & 0 & -a_2 & -b_2 & -a_1 \\ 0 & -a_4 & -a_3 & -a_1 & -b_1 \end{pmatrix}$$

An intermediate B_2 system

Roots

Taking $\Phi = \{\alpha, \beta, \alpha + \beta\}$ we obtain another integrable system with Lax matrix L

The matrix L_+ is defined as above, i.e. the skew-symmetric part of L. Again there is rational integral given by

$$I_{11} = \frac{a_1 a_2 - a_3 b_2}{a_3} \, .$$

Defining the Poisson bracket by
$\{a_1, a_2\} = a_3$,
$\{a_i, b_i\} = -a_i \;\; i = 1, 2$ and
$\{a_1, b_2\} = a_1$ we verify easily that h_2 plays the role of the Hamiltonian and I_{11} is a Casimir. The set $\{h_2, h_4, I_{11}\}$ is an independent set of functions in involution.

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- As in the case of classical and Full Toda there is also an analogous system defined by a Lax matrix which is lower triangular (the Kostant-Toda lattices). We restrict our attention to this version of the systems.
- We show that these Hamiltonian systems are associated to a nilpotent ideal of a Borel subalgebra of a semi-simple Lie algebra $\mathfrak{g}$.
- Since for particular (extreme) choices of the ideal one finds the classical Kostant-Toda lattice or the full Kostant-Toda lattice, associated to $\mathfrak{g}$, we call these Hamiltonian systems Intermediate Toda lattices.

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- We fix an element ε in $\mathfrak{n}_+$, satisfying $\langle \varepsilon \mid [\mathfrak{n}_-, \mathfrak{n}_-] \rangle = 0$, where $\langle \cdot \mid \cdot \rangle$ stands for the Killing form of $\mathfrak{g}$.
- One usually picks for ε a principal nilpotent element of $\mathfrak{n}_+$.

Example in $\mathfrak{g} = sl_N(\mathbf{C})$

$$\varepsilon := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}.$$

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- When $\mathcal{I} = \{0\}$, $M_{\mathcal{I}} = \mathfrak{b}_- + \varepsilon$, which is the phase space of the full Kostant-Toda lattice.
- On the other extreme, taking $\mathcal{I} = [\mathfrak{n}_+ \,, \mathfrak{n}_+]$ the manifold $M_{\mathcal{I}}$ is the phase space of the classical Kostant-Toda lattice. We therefore call $M_{\mathcal{I}}$ the Intermediate Kostant-Toda phase space.

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The endomorphism $R : \mathfrak{g} \to \mathfrak{g}$, defined for all $X \in \mathfrak{g}$ by $R(X) := X_+ - X_-$ is an R-matrix

$$[X, Y]_R := \frac{1}{2}([R(X), Y] + [X, R(Y)]) = [X_+, Y_+] - [X_-, Y_-],$$

for all $X, Y \in \mathfrak{g}$, is a (new) Lie bracket on $\mathfrak{g}$.

Lie-Poisson bracket on $\mathfrak{g}$

$$\{F,G\}(X) = \langle X \mid [(\nabla_X F)_+, (\nabla_X G)_+] \rangle - \langle X \mid [(\nabla_X F)_-, (\nabla_X G)_-] \rangle$$

for every pair of functions F, G on $\mathfrak{g}$ and for all $X \in \mathfrak{g}$.

Poisson Bracket

Proposition

Let $\mathcal{I}$ be a nilpotent ideal of $\mathfrak{b}_+$.

(1) The affine space $M_{\mathcal{I}}$ is a Poisson submanifold of $(\mathfrak{g}, \{\cdot\,,\cdot\})$;

(2) Equipped with the induced Poisson structure, $M_{\mathcal{I}}$ is isomorphic to $(\mathfrak{b}_+/\mathcal{I})^*$, equipped with the canonical Lie-Poisson bracket;

(3) A function F on $M_{\mathcal{I}}$ is a Casimir function if and only if $(\nabla_X \tilde{F})_+ \in \mathcal{I}$ for all $X \in M_{\mathcal{I}}$, where $\tilde{F}$ is an arbitrary extension of F to $\mathfrak{g}$.

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Vector field of the intermediate Kostant-Toda lattice is given by the Lax equation (on $M_{\mathcal{I}}$)

$$\dot{X} = [X_+, X]\,.$$

Nilpotent Ideals- P. Cellini and P. Papi 2000

Properties

Nilpotent Ideals- P. Cellini and P. Papi 2000

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- If $\mathcal{I}$ is a nilpotent ideal of $\mathfrak{b}_+$, then $\mathcal{I}$ is contained in $\mathfrak{n}_+$.

Nilpotent Ideals- P. Cellini and P. Papi 2000

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- Every nilpotent ideal of $\mathfrak{b}_+$ is of this form, for a certain admissible set of roots Φ.
- Thus, the nilpotent ideals of a given Borel subalgebra $\mathfrak{b}_+$ of $\mathfrak{g}$ are parametrized by the family of all subsets Φ of Δ^+, which have the property that if $\alpha \in \Phi$ then every root of the form $\alpha + \beta$, with $\beta \in \Delta^+$, belongs to Φ.

Number of Nilpotent Ideals

Lie Algebra	Number of Positive Roots	Number of Ideals
A_n	$\binom{n+1}{2}$	$\mathcal{C}_{n+1}$
B_n, C_n	n^2	$\binom{2n}{n}$
D_n	$n^2 - n$	$(n+1)\mathcal{C}_n - n\mathcal{C}_{n-1}$
G_2	6	8
F_4	24	105
E_6	36	832
E_7	63	4160
E_8	120	25080

$$\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$$

Formula for counting Nilpotent ideals

The number of ad-nilpotent ideals of $\mathfrak{b}$ is

$$\frac{1}{|W|}\prod_{i=1}^{\ell}(h+m_i+1)=\prod_{i=1}^{\ell}\frac{(h+m_i+1)}{m_i+1}$$

where W is the Coxeter group, h is the Coxeter number and m_i are the exponents.

Height of a root

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$\mathcal{I}_2 = [\mathfrak{n}_+, [\mathfrak{n}_+, \mathfrak{n}_+]]$ and the corresponding affine space $M_{\mathcal{I}_2}$. We call this the Height 2 System

Example

Roots

Consider a Lie algebra of type C_4. Take $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4\}$. It gives rise to a heigh 2 Toda system.
We need five functions to establish integrability.
Since $det(L - \lambda I)$ is an even polynomial of the form

$$\lambda^8 + \sum_{i=0}^{3} f_i \lambda^{2i}$$

we obtain four polynomial integrals

f_0, f_1, f_2, f_3. Using an one-chop we

obtain a characteristic polynomial of the

form $A\lambda^2 + B$. The function $f_4 = B/A$

is the fifth integral.

Lax Matrix

$$L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & a_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_1 & b_2 & a_3 & 1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & b_3 & a_4 & 1 & 0 & 0 & 0 \\ 0 & 0 & c_3 & b_4 & -a_4 & -1 & 0 & 0 \\ 0 & 0 & 0 & c_3 & -b_3 & -a_3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -c_2 & -b_2 & -a_2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -c_1 & -b_1 & -a_1 \end{pmatrix}$$

The function

$$a_1 - a_2 + a_3 - a_4 + \frac{2b_1b_2c_3 + b_1c_2b_4 + b_3b_4c_1}{c_1c_3}$$

is a Casimir.

Stabilizer

The stabilizer of a linear form $\varphi \in \mathfrak{a}^*$ is given by

$$\mathfrak{a}^\varphi := \{x \in \mathfrak{a} \mid \mathrm{ad}_x^* \varphi = 0\} = \{x \in \mathfrak{a} \mid \forall y \in \mathfrak{a},\ \langle \varphi, [x, y] \rangle = 0\}.$$

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Symplectic leaves

Since the symplectic leaves of the canonical Lie-Poisson structure on $\mathfrak{a}^*$ are the coadjoint orbits, the codimension of the symplectic leaf through φ is the dimension of $\mathfrak{a}^\varphi$. It follows that the index of $\mathfrak{a}$ is the codimension of a symplectic leaf of maximal dimension, i.e., the rank of the canonical Lie-Poisson structure on $\mathfrak{a}^*$ is given by $\dim \mathfrak{a} - \mathrm{I}nd(\mathfrak{a})$

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Rank of Lie-Poisson structure

We can use regular linear forms to compute the index of $\mathfrak{a}$, and hence the rank of the canonical Lie-Poisson structure on $\mathfrak{a}^*$.

Let $\mathfrak{a}$ be a subalgebra of a semi-simple complex Lie algebra $\mathfrak{g}$. Suppose that φ is a linear form on $\mathfrak{a}$, such that $\mathfrak{a}^{\varphi}$ is a commutative Lie algebra composed of semi-simple elements. Then φ is regular, so that the index of $\mathfrak{a}$ is given by $\dim \mathfrak{a}^{\varphi}$.

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Proof.

A linear form $\varphi \in \mathfrak{a}^*$ is said to be *stable* if there exists a neighborhood U of φ in $\mathfrak{a}^*$ such that for every $\psi \in U$, the stabilizer $\mathfrak{a}^\psi$ is conjugate to $\mathfrak{a}^\varphi$, with respect to the adjoint group of $\mathfrak{a}$. Every stable linear form is regular. φ is stable if and only if $[\mathfrak{a}, \mathfrak{a}^\varphi] \cap \mathfrak{a}^\varphi = \{0\}$. The latter equality holds when $\mathfrak{a}^\varphi$ is a commutative Lie algebra composed of semi-simple elements. Thus, φ is stable, hence regular.

□

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Consider the linear form φ on $\mathfrak{b}_+$, defined for $Z \in \mathfrak{b}_+$ by $\langle \varphi, Z \rangle := \langle X \mid Z \rangle$, where X is defined by

$$X := \delta_\ell X_{-\alpha_\ell} + \sum_{i=1}^{\ell-1} X_{-\alpha_i - \alpha_{i+1}},$$

with $\delta_\ell := 1$ if ℓ is odd and $\delta_\ell := 0$ otherwise. Denote by $\bar{\varphi}$ the induced linear form on $\mathfrak{b}_+/\mathcal{I}_2$.

(1) $\bar{\varphi}$ is a regular linear form on $\mathfrak{b}_+/\mathcal{I}_2$;
(2) $\dim(\mathfrak{b}_+/\mathcal{I}_2)^{\bar{\varphi}} = 1 - \delta_\ell$;
(3) The index of $\mathfrak{b}_+/\mathcal{I}_2$ is 1 if the rank ℓ of $\mathfrak{g}$ is even and is 0 otherwise.

Integrability

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Hamiltonian is part of a family of s independent functions in involution, where s is related to the dimension and the rank of the Poisson manifold $M_{\mathcal{I}_2}$ by the formula

$$\dim M_{\mathcal{I}_2} = \frac{1}{2}\mathsf{Rk}\, M_{\mathcal{I}_2} + s.$$

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As we will see, they can be constructed by restricting certain chop-type integrals, except for the case of C_ℓ where another integral (Casimir) is needed.

A_l

We first consider $\mathfrak{g} = sl_{\ell+1}(\mathbf{C})$, the Lie algebra of traceless matrices of size $N = \ell + 1$, taking for $\mathfrak{h}$, Π and ε the standard choices, as before. A general element of $\mathcal{M}_{\mathcal{I}_2}$ is then of the form

$$X = \begin{pmatrix} a_1 & 1 & 0 & \dots & \dots & 0 \\ b_1 & a_2 & 1 & \ddots & & \vdots \\ c_1 & b_2 & a_3 & 1 & \ddots & \vdots \\ 0 & c_2 & b_3 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & c_{\ell-1} & b_\ell & a_{\ell+1} \end{pmatrix},$$

with $\sum_{i=1}^{\ell+1} a_i = 0$.

The 1-chop matrix of X is given by

$$(X - \lambda \, \mathrm{Id}_{\ell+1})_1 = \begin{pmatrix} b_1 & a_2^\lambda & 1 & 0 & \dots & 0 \\ c_1 & b_2 & a_3^\lambda & 1 & \ddots & \vdots \\ 0 & c_2 & b_3 & \ddots & \ddots & 0 \\ \vdots & \ddots & c_3 & b_4 & \ddots & 1 \\ \vdots & & \ddots & \ddots & \ddots & a_\ell^\lambda \\ 0 & \dots & \dots & 0 & c_{\ell-1} & b_\ell \end{pmatrix},$$

where a_i^λ is a shorthand for $a_i - \lambda$. We also use the matrix $X(\lambda, \alpha)$, defined by

$$X(\lambda,\alpha) = \begin{pmatrix} b_1 & a_2^{\lambda} & \alpha_{13} & \dots & \dots & \alpha_{1\ell} \\ c_1 & b_2 & a_3^{\lambda} & \alpha_{24} & & \vdots \\ 0 & c_2 & b_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & c_3 & b_4 & \ddots & \alpha_{\ell-2,\ell} \\ \vdots & & \ddots & \ddots & \ddots & a_{\ell}^{\lambda} \\ 0 & \dots & \dots & 0 & c_{\ell-1} & b_{\ell} \end{pmatrix}.$$

The polynomials $\det(X - \lambda\,\mathrm{Id}_{\ell+1})_1$ and $\det X(\lambda, \alpha)$ have degree $d := [\frac{\ell}{2}]$ in λ;

Integrability

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Proposition

The functions $H_1, \ldots, H_\ell, I_1, \ldots, I_{[\ell/2]}$ form a set of constants of motion in involution. When ℓ is odd, the rank of the underlying Poisson manifold $M_{\mathcal{I}_2}$ is maximal, while the rank is $\dim M_{\mathcal{I}_2} - 1$ otherwise, with $I_{\ell/2}$ as a Casimir function.

B_l

Chops

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- A Lie algebra of type B_ℓ can be realized as the Lie algebra $\mathfrak{g}$ of all square matrices of size $N = 2\ell + 1$, satisfying $XJ + JX^t = 0$, where J is the matrix of size $2\ell + 1$, all of whose entries are zero, except for the entries on the anti-diagonal, which are all equal to one.

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- $\det(X - \lambda\,\mathrm{Id}_{\ell+1}) = (-1)^N \det(X + \lambda\,\mathrm{Id}_{\ell+1})$, so that the characteristic polynomial is an odd polynomial in λ.

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- $\det(X - \lambda \operatorname{Id}_{\ell+1}) = (-1)^N \det(X + \lambda \operatorname{Id}_{\ell+1})$, so that the characteristic polynomial is an odd polynomial in λ.
- The 1-chop matrix X_1 satisfies the same relation $X_1 J + JX_1^t = 0$, so that its determinant is an even polynomial in λ.

Integrability

Lax matrix

$$\begin{pmatrix}
a_1 & 1 & & & & & & & & \\
b_1 & \ddots & \ddots & & & & & & & \\
c_1 & \ddots & \ddots & 1 & & & & & & \\
 & \ddots & b_{n-1} & a_n & 1 & & & & & \\
 & & c_{n-1} & b_n & 0 & -1 & & & & \\
 & & & 0 & -b_n & -a_n & \ddots & & & \\
 & & & & -c_{n-1} & -b_{n-1} & \ddots & \ddots & & \\
 & & & & & \ddots & \ddots & & -1 \\
 & & & & & -c_1 & -b_1 & -a_1 &
\end{pmatrix}$$

In this case $N = 2\ell + 1$, the 1-chop polynomial is even, so the 1-chop polynomial is degree ℓ when ℓ is even and of degree $\ell - 1$ when ℓ is odd. This yields $\frac{\ell}{2}$ integrals when ℓ is even and $\frac{\ell-1}{2}$ when ℓ is odd. Therefore the number of integrals is correct in each case.

C_l

Chops

C_l

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- A Lie algebra of type C_ℓ can be realized as the Lie algebra $\mathfrak{g}$ of all square matrices of size $N = 2\ell$, satisfying $XJ + JX^t = 0$, where J is the matrix of size 2ℓ, given by

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c_1 & b_2 & \ddots & 1 & & & & & \\
 & \ddots & \ddots & a_n & 1 & & & & \\
 & & c_{n-1} & b_n & -a_n & -1 & & & \\
 & & & c_{n-1} & -b_{n-1} & \ddots & \ddots & & \\
 & & & & -c_{n-2} & \ddots & \ddots & \ddots & \\
 & & & & & \ddots & \ddots & & -1 \\
 & & & & & -c_1 & -b_1 & -a_1 &
\end{pmatrix}.$$

In this case $N = 2\ell$, the 1-chop polynomial is even, so we get $\frac{l}{2} - 1$ integrals from the 1-chop when l is even and $\frac{l-2}{2}$ integrals when l is odd. Therefore the odd case gives the correct number of integrals. For the even case there exists a Casimir function which does not arise from the method of chopping.

C_l Casimir

Definition

The Casimir f has the form

$$f = A + \frac{B}{C} \ ,$$

where

$$A = \sum_{i=1}^{\ell-1} (a_i - a_{i+1}) \ ,$$

$$C = \prod_{i=1}^{\ell-1} c_{2i-1} \ .$$

and

$$B = \sum_{i,j} d_{ij} m_{ij} \ .$$

The term m_{ij} in B is determined as follows: We associate the variables $b_1, b_2, \ldots, b_l$ to the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_l$ and the variables $c_1, c_2, \ldots, c_{l-1}$ to the height 2 roots $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \ldots, \alpha_{l-1} + \alpha_l$.
Take simple roots α_i, α_j (with corresponding variables b_i, b_j) such that i is odd and j is even. The remaining variables correspond to the height two roots $\alpha_k + \alpha_{k+1}$ where $k \neq i, i-1$, $k \neq j, j-1$. The term m_{ij} is a product of b_i, b_j and $\frac{l-1}{2}$ c variables.

The coefficient d_{ij} is 2 if m_{ij} includes the term c_{l-1}

(corresponding to the root $\alpha_{l-1} + \alpha_l$), and is equal to 1 otherwise.

Example: $l = 6$

$$f = A + \frac{B}{C}$$

$$A = a_1 - a_2 + a_3 - a_4 + a_5 - a_6$$

$$B = b_5 b_6 c_1 c_3 + 2 b_1 b_4 c_2 c_5 + b_3 b_6 c_1 c_4 + 2 b_1 b_2 c_3 c_5 + 2 b_3 b_4 c_1 c_5 + b_1 b_6 c_2 c_4$$

$$C = c_1 c_3 c_5$$

D_ℓ

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- A Lie algebra of type D_ℓ can be realized as the Lie algebra $\mathfrak{g}$ of all square matrices of size $N = 2\ell$, satisfying $XJ + JX^t = 0$, where J is the matrix of size 2ℓ, given by

$$J = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix} .$$

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- As in the case of C_ℓ the characteristic polynomial is an even polynomial. On the other hand the 1-chop polynomial is odd so the degree of this polynomial is $\ell - 1$ when ℓ is even. But when ℓ is odd the degree of the 1-chop polynomial is again ℓ. This gives $\frac{\ell}{2} - 1$ integrals when ℓ is even and $\frac{\ell-1}{2}$ integrals when ℓ is odd. In the even case we need an extra function i.e. a Casimir but at this point we do not have an explicit formula. There is no stable form in this case, but we can produce a form which gives a lower bound for the rank and this lower bound is good enough, once we have the Casimir.

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  (b) **Lotka** to investigate autocatalytic chemical reactions.

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- e.g. in game dynamics: replicator equations $\leftrightarrow$ Lotka-Volterra (LV)

KM system

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- Our generalization is different from the one of Bogoyavlensky.

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$$L = \begin{pmatrix} x_1 & 0 & \sqrt{x_1 x_2} & 0 & \dots & & 0 \\ 0 & x_1 + x_2 & 0 & \sqrt{x_2 x_3} & & & \vdots \\ \sqrt{x_1 x_2} & 0 & x_2 + x_3 & & \ddots & & \\ 0 & \sqrt{x_2 x_3} & & & & & \\ \vdots & & \dots & & & & \sqrt{x_{n-1} x_n} \\ & & & & & x_{n-1} + x_n & 0 \\ & & & & \sqrt{x_{n-1} x_n} & 0 & x_n \end{pmatrix}$$

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- for the Lotka-Volterra system is defined by

$$\{x_i, x_j\} = a_{ij} x_i x_j, \ \ i, j = 1, 2, \ldots, n \ .$$

- It is clear that the skew-symmetric matrix $A = (a_{ij})$ determines the bracket.

LV systems: Properties

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- For a root system of type A_n we obtain the KM system.

Second approach: From the Dynkin diagram

It works for ADE Dynkin diagrams

$$\dot{x}_i = x_i \sum_{j=1}^{\ell} m_{ij} x_j$$

where the skew-symmetric matrix m_{ij} for $i < j$ is defined to be $m_{ij} = 1$ if vertex i is connected with vertex j and 0 otherwise

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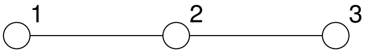
- There are several inequivalent ways to label a graph and therefore the association between graphs and Lotka-Volterra systems is not always a bijection.
- The number of distinct labellings of a given unlabeled simple graph G on n vertices is known to be

$$\frac{n!}{|\mathrm{aut}\,(G)|} .$$

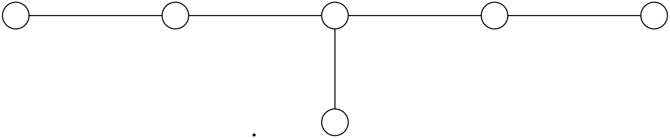
A_3 Diagram



Consider a Dynkin diagram with graph A_3. We label the vertices from left to right. To define $\dot{x}_1$ we note that vertex 1 is joined only with vertex 2. Therefore we include a term x_1x_2. We define $m_{13} = 0$ since vertex 1 is not connected with vertex 3. Similarly we define $m_{23} = 1$ since vertex 2 is connected with vertex 3. Therefore we obtain the KM system

$$\begin{aligned} \dot{x}_1 &= x_1x_2 \\ \dot{x}_2 &= -x_1x_2 + x_2x_3 \\ \dot{x}_3 &= -x_2x_3. \end{aligned}$$

E_6 Diagram



$$\begin{aligned}
\dot{x}_1 &= x_1 x_2 \\
\dot{x}_2 &= x_2(-x_1 + x_3) \\
\dot{x}_3 &= x_3(-x_2 + x_4 + x_5) \\
\dot{x}_4 &= -x_3 x_4 \\
\dot{x}_5 &= x_5(-x_3 + x_6) \\
\dot{x}_6 &= -x_5 x_6 \, .
\end{aligned}$$

The associated Poisson structure is symplectic. Therefore to prove integrability one needs another two constants of motion besides the Hamiltonian.

D_4 Dynkin Diagram

Equations from the Dynkin diagram of the simple Lie algebra of type D_4

$$\begin{aligned}
\dot{x}_1 &= x_1x_2 \\
\dot{x}_2 &= -x_1x_2 + x_2x_3 + x_2x_4 \\
\dot{x}_3 &= -x_2x_3 \\
\dot{x}_4 &= -x_2x_4 \, .
\end{aligned}$$

Lax pair

$$L = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & x_2 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & x_3 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & x_4 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & x_4 & -x_3 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -x_2 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -x_1 & 0
\end{pmatrix}$$

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$4H_2 = x_1+x_2+x_3+x_4$ $\qquad$ $4H_4 = x_1^2+x_2^2+x_3^2+x_4^2+2x_1x_2+2x_2x_3+2x_2x_4+2x$

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There are also two Casimirs $F_1 = x_1x_4$ and $F_2 = x_1x_3$.
We can find the Casimirs by computing the kernel of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \; .$$

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The two eigenvectors with eigenvalue 0 are $(1, 0, 0, 1)$ and $(1, 0, 1, 0)$. We obtain the two Casimirs $F_1 = x_1^1x_2^0x_3^0x_4^1 = x_1x_4$ and $F_2 = x_1^1x_2^0x_3^1x_4^0 = x_1x_3$.

An alternative method to define the systems is the following. Let $\tilde{A} = 2I - C$ be the Coxeter adjacency matrix. Decompose $\tilde{A} = A + B$ where $A = (a_{ij})$ is the skew-symmetric part of $\tilde{A}$ and B its lower triangular part. Define the Lotka-Volterra system using the formula

$$\dot{x}_i = \sum_{j=1}^{n} a_{ij} x_i x_j, \;\; i = 1, 2, \ldots, n \;\; .$$

This method can be used to define Lotka-Volterra systems for any complex simple Lie algebra (including B_n, C_n, G_2 and F_4).

Example B_3

The Cartan matrix is given by

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} .$$

Since

$$2I - C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

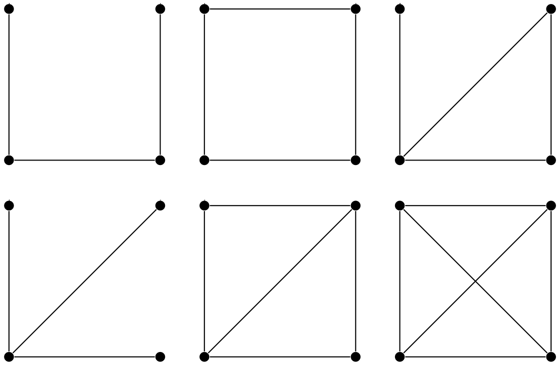
we may define a B_3 Lotka-Volterra system as follows:

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 \\ \dot{x}_2 &= -x_1 x_2 + 2x_2 x_3 \\ \dot{x}_3 &= -2x_2 x_3 . \end{aligned}$$

The Casimir for this system is $F = x_1^2 x_3$.

Connected Graphs on four vertices

It is well-known that there are 6 connected simple graphs on four vertices. They are given in the following figure.



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- Let $\Pi = \{\alpha_1, \dots \alpha_\ell\}$.
- Let $X_{\alpha_1}, \cdots, X_{\alpha_\ell}$ be the corresponding root vectors in $\mathfrak{g}$.
- Define: $$L = \sum_{\alpha_i \in \Pi} a_i X_{\alpha_i}.$$

Finding the matrix B:

- For each i, j form $[X_{\alpha_i}, X_{\alpha_j}]$.
  - If $\alpha_i + \alpha_j$ is a root then
  - include a term of the form $\pm a_i a_j [X_{\alpha_i}, X_{\alpha_j}]$ in B.

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- For a root system of type A_n we obtain the KM system.

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The Lax pair is equivalent to the following equations of motion:

$$\dot{a_1} = a_1a_2^2 - a_1a_4^2$$
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- The two eigenvectors with eigenvalue 0 are:
  - $(1, 0, 1, 0)$ and $(1, 1, 0, 1)$.
- We obtain the two Casimirs:
  - $F_1 = x_1^1 x_2^0 x_3^1 x_4^0 = x_1x_3$ and $F_2 = x_1^1 x_2^1 x_3^0 x_4^1 = x_1x_2x_4$.

Consider the generalized Lotka-Volterra system defined by the Lax matrix

$$L = \begin{pmatrix} 0 & a_1 & 0 & a_5 & 0 \\ a_1 & 0 & a_2 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 \\ a_5 & 0 & a_3 & 0 & a_4 \\ 0 & 0 & 0 & a_4 & 0 \end{pmatrix}$$

which corresponds to the subset $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2 + \alpha_3\}$. Define the matrix B to be

$$\begin{pmatrix} 0 & 0 & a_1a_2 + a_3a_5 & 0 & a_4a_5 \\ 0 & 0 & 0 & a_2a_3 + a_1a_5 & 0 \\ -a_1a_2 - a_3a_5 & 0 & 0 & 0 & a_3a_4 \\ 0 & -a_2a_3 - a_1a_5 & 0 & 0 & 0 \\ -a_4a_5 & 0 & -a_3a_4 & 0 & 0 \end{pmatrix}.$$

The Hamiltonian of the system is $H = \frac{1}{2}\left(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2\right)$ and the Poisson matrix (of rank 4) is

$$\begin{pmatrix} 0 & a_1a_2 & 2a_2a_5 & 0 & a_1a_5 \\ -a_1a_2 & 0 & a_2a_3 & 0 & 0 \\ -2a_2a_5 & -a_2a_3 & 0 & a_3a_4 & -a_3a_5 \\ 0 & 0 & -a_3a_4 & 0 & -a_4a_5 \\ -a_1a_5 & 0 & a_3a_5 & a_4a_5 & 0 \end{pmatrix}.$$

The system is integrable with constants of motion
$H = \frac{1}{2}\left(a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2\right)$ and

$$F = \operatorname{tr}\left(\frac{L^4}{4}\right) = \frac{1}{2}a_1^4 + a_1^2a_5^2 + \frac{1}{2}a_5^4 + a_1^2a_2^2 + 2a_1a_5a_2a_3 + a_3^2a_5^2 + a_4^2a_5^2 + \frac{1}{2}a_2^4 + a_2^2a_3^2$$

The Casimir of the system is $C = a_2^2 - \frac{a_1a_2a_3}{a_5}$ and may be obtained by the method chopping as follows.

We have

$$x \cdot I_5 - L = \begin{pmatrix} x & -a_1 & 0 & -a_5 & 0 \\ -a_1 & x & -a_2 & 0 & 0 \\ 0 & -a_2 & x & -a_3 & 0 \\ -a_5 & 0 & -a_3 & x & -a_4 \\ 0 & 0 & 0 & -a_4 & x \end{pmatrix}$$

and the one-chopped matrix is

$$\begin{pmatrix} -a_1 & x & -a_2 & 0 \\ 0 & -a_2 & x & -a_3 \\ -a_5 & 0 & -a_3 & x \\ 0 & 0 & 0 & -a_4 \end{pmatrix}$$

with determinant $a_4a_5x^2 + a_1a_2a_3a_4 - a_2^2a_4a_5$. Dividing the constant term of this polynomial by the leading term a_4a_5 we obtain the Casimir C.

Example

The matrix L is given by

$$L = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & a_8 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 & 0 & a_9 & 0 \\ 0 & a_2 & 0 & a_3 & 0 & 0 & 0 & a_{10} \\ 0 & 0 & a_3 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & a_5 & 0 & 0 \\ a_8 & 0 & 0 & 0 & a_5 & 0 & a_6 & 0 \\ 0 & a_9 & 0 & 0 & 0 & a_6 & 0 & a_7 \\ 0 & 0 & a_{10} & 0 & 0 & 0 & a_7 & 0 \end{pmatrix},$$

The corresponding system is given by:

$$
\begin{aligned}
\dot{a}_1 &= a_1a_2^2 + a_1a_9^2 - a_1a_8^2,\\
\dot{a}_2 &= a_2a_3^2 - a_1^2a_2 + a_2a_{10}^2 - a_2a_9^2,\\
\dot{a}_3 &= a_3a_4^2 - a_2^2a_3 - a_3a_{10}^2,\\
\dot{a}_4 &= a_4a_5^2 - a_3^2a_4,\\
\dot{a}_5 &= a_5a_6^2 - a_4^2a_5 + a_5a_8^2,\\
\dot{a}_6 &= a_6a_7^2 - a_5^2a_6 + a_6a_9^2 - a_6a_8^2,\\
\dot{a}_7 &= -a_6^2a_7 + a_7a_{10}^2 - a_7a_9^2,\\
\dot{a}_8 &= a_1^2a_8 + a_6^2a_8 - a_5^2a_8 + 2\,a_1a_6a_9,\\
\dot{a}_9 &= a_2^2a_9 - a_1^2a_9 + a_7^2a_9 - a_6^2a_9 + 2\,a_2a_7a_{10} - 2\,a_1a_6a_8,\\
\dot{a}_{10} &= a_3^2a_{10} - a_2^2a_{10} - a_7^2a_{10} - 2\,a_2a_7a_9.
\end{aligned}
$$

It is a Hamiltonian system with Poisson structure determined by the Poisson matrix

$$\Big(\quad 0 \quad a_1a_2 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -a_1a_8 \quad a$$

THANK YOU!