New generalizations of the (2+1)-dimensional k-constrained KP hierarchies

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Program of this talk

- Symmetry reductions of the KP hierarchy (k-constrained KP hierarchy)
- (2+1)-dimensional k-constrained KP hierarchies
- Extensions of the (2+1)-dimensional k-constrained KP hierarchies
- A solution generating method
- Conclusions

k-constrained KP hierarchy

Consider a microdifferential Lax operator:

$$L := D + \sum_{i=1}^{\infty} U_i D^{-i} \qquad D := \frac{\partial}{\partial x}$$

with functions $U_i = U_i(t_1, t_2, t_3, \dots)$, which depend on an arbitrary (finite) number of independent variables $t_1 := x,\ t_2,\ t_3,\ \dots$
KP hierarchy is the following family of Lax equations for L:

$$\alpha_i L_{t_i} = [B_i, L] \qquad \alpha_i \in \mathbb{C} \qquad i \in \mathbb{N}$$

$B_i := (L^i)_{\geq 0}$ is a differential part of the i-th power of L. From the latter Zakharov-Shabat equations arise: $L_{t_m t_n} = L_{t_n t_m} \Rightarrow$

$$[\alpha_n \partial_{t_n} - B_n, \alpha_m \partial_{t_m} - B_m] = \alpha_m B_{n,t_m} - \alpha_n B_{m,t_n} + [B_n, B_m] = 0$$

If $n = 2,\ m = 3$ the latter is equivalent to the KP eq. $(U_1 := U)$:

$$(\alpha_3 U_{t_3} - \frac{1}{4} U_{xxx} - 3UU_x)_x = \frac{3}{4}\alpha_2^2 U_{t_2 t_2}.$$

k-constrained KP hierarchy

Consider the symmetry reduction of the KP hierarchy:

$$(L^k)_{<0} = \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top$$

where $\mathrm{Mat}_{l\times l}(\mathbb{C}) \ni \mathcal{M}_0$ is a constant matrix, and functions $\mathbf{q} = (q_1, ..., q_l)$, $\mathbf{r} = (r_1, ..., r_l)$ are fixed solutions of the system:

$$\alpha_n \mathbf{q}_{t_n} = B_n\{\mathbf{q}\} \qquad \alpha_n \mathbf{r}_{t_n} = -B_n^\tau\{\mathbf{r}\} \qquad B_n := (L^n)_{\geq 0}$$

Reduced flows admit the following Lax representation

$$[L_k, M_n] = 0 \qquad L_k := L^k = B_k + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top \qquad M_n = \alpha_n \partial_{t_n} - B_n.$$

The latter is equivalent to the $(1+1)$-dimensional integrable systems for functional coefficients U_i and functions $\mathbf{q}$, $\mathbf{r}$:

$$U_{it_n} = P_{in}[U_1, U_2, ..., U_{k-1}, \mathbf{q}, \mathbf{r}] \qquad i = \overline{1, k-1}$$
$$\alpha_n \mathbf{q}_{t_n} = B_n[U_i, \mathbf{q}, \mathbf{r}]\{\mathbf{q}\} \qquad \alpha_n \mathbf{r}_{t_n} = -B_n^\tau[U_i, \mathbf{q}, \mathbf{r}]\{\mathbf{r}\}$$

Examples

1. $k=1, n=2: \;\; L_1 = D + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^*, \quad M_2 = i\partial_{t_2} - D^2 - 2\mathbf{q}\mathcal{M}_0\mathbf{q}^*$ with $\mathcal{M}_0^* = \mathcal{M}_0$. Lax equation $[L_1, M_2] = 0$ is equivalent to the NLS:

$$i\mathbf{q}_{t_2} = \mathbf{q}_{xx} + 2\left(\mathbf{q}\mathcal{M}_0\mathbf{q}^*\right)\mathbf{q}.$$

2. $k=2, n=2: \;\; L_2 = D^2 + 2u + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^*, \quad M_2 = i\partial_{t_2} - D^2 - 2u$ with $\mathcal{M}_0^* = -\mathcal{M}_0$. Equation $[L_2, M_2] = 0$ is equivalent to the Yajima-Oikawa system:

$$i\mathbf{q}_{t_2} = \mathbf{q}_{xx} + 2u\mathbf{q}, \qquad i u_{t_2} = \left(\mathbf{q}\mathcal{M}_0\mathbf{q}^*\right)_x.$$

3. $k=2, n=3: \;\; L_2 = D^2 + 2u + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^*,$
   $M_3 = \partial_{t_3} - D^3 - 3uD - \frac{3}{2}(u_x + \mathbf{q}\mathcal{M}_0\mathbf{q}^*), \qquad \mathcal{M}_0^* = -\mathcal{M}_0$
   $[L_2, M_3] = 0$ is equivalent to the KdV with self-consistent sources:

$$\mathbf{q}_{t_3} = \mathbf{q}_{xxx} + 3u\mathbf{q}_x + \frac{3}{2}u_x\mathbf{q} + \frac{3}{2}\mathbf{q}\mathcal{M}_0\mathbf{q}^*\mathbf{q},$$
$$u_{t_3} = \frac{1}{4}u_{xxx} + 3uu_x + \frac{3}{4}\left(\mathbf{q}_x\mathcal{M}_0\mathbf{q}^* - \mathbf{q}\mathcal{M}_0\mathbf{q}_x^*\right)_x.$$

Examples

4. Vector generalization of the modified KdV equation

$$\mathbf{q}_{t_3} = \mathbf{q}_{xxx} + 3\left(\mathbf{q}\mathcal{M}_0\mathbf{q}^*\right)\mathbf{q}_x + 3\left(\mathbf{q}_x\mathcal{M}_0\mathbf{q}^*\right)\mathbf{q} \qquad \mathcal{M}_0^* = \mathcal{M}_0.$$

5. Generalization of the Boussinesq equation:

$$\begin{aligned} &3\alpha_2^2 u_{t_2 t_2} = \left(-u_{xx} - 6u^2 + 4(\mathbf{q}\mathcal{M}_0\mathbf{q}^*)\right)_{xx} \qquad \mathcal{M}_0^* = \mathcal{M}_0 \\ &\alpha_2\mathbf{q}_{t_2} - \mathbf{q}_{xx} - 2u\mathbf{q} = 0 \end{aligned}$$

6. Vector generalization of the Drinfeld-Sokolov-Wilson system:

$$\begin{aligned} &\mathbf{q}_{t_3} = \mathbf{q}_{xxx} + 3u\mathbf{q}_x + \tfrac{3}{2}u_x\mathbf{q} \qquad \mathbf{q} = \bar{\mathbf{q}} \qquad \mathcal{M}_0^* = \mathcal{M}_0 = \bar{\mathcal{M}}_0 \\ &u_{t_3} = (\mathbf{q}\mathcal{M}_0\mathbf{q}^\top)_x \end{aligned}$$

k-constrained KP hierarchy

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(2+1)-dimensional k-cKP hierarchies

Introduce (2+1)-dimensional generalizations of the k-cKP hierarchy:

$$[L_k, M_n] = 0 \qquad L_k = \alpha\partial_y - \sum_{i=0}^{k} u_i D^i - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top$$

$$M_n = \alpha_n\partial_{t_n} - \sum_{j=0}^{n} v_j D^j \qquad M_n\{\mathbf{q}\} = 0 \qquad M_n^\tau\{\mathbf{r}\} = 0$$

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Examples: Davey-Stewartson (DS-III) and (2+1)-dimensional Yajima-Oikawa systems

1. $k = 1, n = 2$:

$$L_1 = \partial_y - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^* \qquad M_2 = i\partial_{t_2} - D^2 - 2u$$

with $u = u(x, y, t_2) = \bar{u}(x, y, t_2)$, $\mathcal{M}_0 = \mathcal{M}_0^*$.
Equation $[L_1, M_2] = 0$ is equivalent to the DS-III system:

$$i\mathbf{q}_{t_2} = \mathbf{q}_{xx} - 2u\mathbf{q} \qquad u_y = (\mathbf{q}\mathcal{M}_0\mathbf{q}^*)_x$$

2. $k = 2, n = 2$:

$$L_2 = i\partial_y - D^2 - 2u - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^* \qquad M_2 = i\partial_{t_2} - D^2 - 2u$$

with $\mathcal{M}_0 = -\mathcal{M}_0^*$, $u = \bar{u}$.
Equation $[L_2, M_2] = 0$ is equivalent to the (2+1)-dimensional generalization of the Yajima-Oikawa system:

$$iu_{t_2} = iu_y + (\mathbf{q}\mathcal{M}_0\mathbf{q}^*)_x \qquad i\mathbf{q}_{t_2} = \mathbf{q}_{xx} + 2u\mathbf{q}$$

Examples: KP equation with self-consistent sources

3. $k = 2, n = 3$:

$$L_2 = i\partial_y - D^2 - 2u - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^*,$$
$$M_3 = \partial_{t_3} - D^3 - 3uD - \frac{3}{2}\left(u_x + iD^{-1}\{u_y\} + \mathbf{q}\mathcal{M}_0\mathbf{q}^*\right),$$

with $\mathcal{M}_0^* = -\mathcal{M}_0,\ u = \bar{u}$. Equation $[L_2, M_3] = 0$ is equivalent to the KP equation with self-consistent sources:

$$\mathbf{q}_{t_3} = \mathbf{q}_{xxx} + 3u\mathbf{q}_x + \frac{3}{2}\left(u_x + iD^{-1}\{u_y\} + \mathbf{q}\mathcal{M}_0\mathbf{q}^*\right)\mathbf{q},$$
$$\Big[u_{t_3} - \frac{1}{4}u_{xxx} - 3uu_x + \frac{3}{4}\left(\mathbf{q}\mathcal{M}_0\mathbf{q}_x^* - \mathbf{q}_x\mathcal{M}_0\mathbf{q}^*\right)_x - $$
$$- \frac{3}{4}i\left(\mathbf{q}\mathcal{M}_0\mathbf{q}^*\right)_y\Big]_x = -\frac{3}{4}u_{yy}.$$

Examples: (2+1)-dimensional Drinfel'd-Sokolov-Wilson system

4. $k = 3, n = 3$:

$$L_3 = \partial_{\tau_3} - D^3 - u_1 D - \tfrac{1}{2} u_{1,x} - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{q}^\top$$
$$M_3 = \partial_{t_3} - D^3 - u_1 D - \tfrac{1}{2} u_{1,x}$$

With the additional reduction $u_1 = \bar{u}_1 := u$, $\mathcal{M}_0 = \mathcal{M}_0^* = \bar{\mathcal{M}}_0$ Lax equation $[L_3, M_3] = 0$ is equivalent to the following (2+1)-dimensional generalization of the Drinfel'd-Sokolov-Wilson equation:

$$\mathbf{q}_{t_3} = \mathbf{q}_{xxx} + u\mathbf{q}_x + \frac{1}{2}u_x\mathbf{q} \qquad u_{t_3} = u_{\tau_3} + 3(\mathbf{q}\mathcal{M}_0\mathbf{q}^\top)_x.$$

New extensions of the (2+1)-k-cKP hierarchies

We introduce the following generalizations of
the (2+1)-dimensional k-constrained KP hierarchies:

$$L_k = \beta_k \partial_{\tau_k} - \sum_{j=0}^{k} u_j D^j - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top$$
$$M_{n,l} = \alpha_n \partial_{t_n} - \sum_{i=0}^{n} v_i D^i - c_l \sum_{j=0}^{l} \mathbf{q}[j]\mathcal{M}_0 D^{-1}\mathbf{r}^\top[l-j], \quad l = 1, \dots$$
$$M_{n,l}\{\mathbf{q}\} = c_l (L_k)^{l+1}\{\mathbf{q}\} \qquad M_{n,l}^{\tau}\{\mathbf{r}\} = c_l (L_k^{\tau})^{l+1}\{\mathbf{r}\}$$
$$u_j = u_j(x, \tau_k, t_n), \quad v_i = v_i(x, \tau_k, t_n), \quad \alpha_n, \beta_k, c_l \in \mathbb{C}$$

where $\mathcal{M}_0$ is $M \times M$ constant matrix, u_j and v_i are $N \times N$ matrix functions; $\mathbf{q}$ and $\mathbf{r}$ are $N \times M$ matrix functions respectively; $\mathbf{q}[j]$ and $\mathbf{r}[j]$ have the form: $\mathbf{q}[j] := (L_k)^j\{\mathbf{q}\}$, $\mathbf{r}^\top[j] := ((L_k^\tau)^j\{\mathbf{r}\})^\top$.

New extensions of the (2+1)-k-cKP hierarchies

Proposition

Lax equation $[L_k, M_{n,l}] = 0$ is equivalent to the system:

$[L_k, M_{n,l}]_{\geq 0} = 0$

$M_{n,l}\{\mathbf{q}\} = c_l(L_k)^{l+1}\{\mathbf{q}\} \qquad M^{\tau}_{n,l}\{\mathbf{r}\} = c_l(L^{\tau}_k)^{l+1}\{\mathbf{r}\}$

In case $k=0,\ n=2,\ l=1$ Lax equation $[L_k, M_{n,l}] = 0$ is equivalent to the following matrix generalization of the DS-system $(y := \tau_0)$

$$\begin{aligned}
\alpha_2 \mathbf{q}_{t_2} &= c\mathbf{q}_{xx} + c_1\mathbf{q}_{yy} + cv_0\mathbf{q} + c_1\mathbf{q}\mathcal{M}_0 S \\
-\alpha_2 \mathbf{r}^{\top}_{t_2} &= c\mathbf{r}^{\top}_{xx} + c_1\mathbf{r}^{\top}_{yy} + c\mathbf{r}^{\top} v_0 + c_1 S\mathcal{M}_0\mathbf{r}^{\top} \\
v_{0y} &= -2(\mathbf{q}\mathcal{M}_0\mathbf{r}^{\top})_x \qquad S_x = -2(\mathbf{r}^{\top}\mathbf{q})_y
\end{aligned}$$

where $\alpha_2, c, c_1 \in \mathbb{C}$, v_0 and S are $N\times N$ and $M\times M$ matrix functions; $\mathbf{q}$ and $\mathbf{r}$ are $N\times M$ matrix functions respectively.

Examples

New extensions of the (2+1)-k-cKP hierarchies also contain:

- (2+1)-dimensional mKdV equation

$$q_{t_3} - q_{xxx} - q_{yyy} + 3q_x \int |q|_x^2 dy + 3\mu q_y \int |q|_y^2 dx +$$
$$+3q \int (\bar{q} q_y)_y dx + 3q \int (q_x q)_x dy = 0.$$

- Nizhnik equation

$$u_{t_3} - u_{xxx} - u_{yyy} + 3\partial_x \left\{ \left(\int u_x dy \right) u \right\} + 3\partial_y \left\{ u \left(\int u_y dx \right) \right\} = 0$$

- Matrix (2+1)-dimensional generalizations of Yajima-Oikawa, Drinfel'd-Sokolov systems, new extensions of KP equations with self-consistent sources.

Solution generating method for (2+1)-dimensional k-cKP hierarchies and their extensions

Theorem. Let matrices of functions φ and ψ satisfy $L\{\varphi\} = \varphi\Lambda$, $L^{\tau}\{\psi\} = \psi\tilde{\Lambda}$ with constant matrices Λ, $\tilde{\Lambda}$ and the following operator

$$L := \alpha\partial_t - \sum_{i=0}^{n} u_i D^i + \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^{\top}, \ \alpha \in \mathbb{C}$$

Construct the binary Darboux transformation (BDT):

$$W = I - \varphi\left(C + D^{-1}\{\psi^{\top}\varphi\}\right)^{-1} D^{-1}\psi^{\top}$$

with some constant matrix C. Then:

$$\hat{L} := WLW^{-1} = \alpha\partial_t - \sum_{i=0}^{n} \hat{u}_i D^i + \hat{\mathbf{q}}\mathcal{M}_0 D^{-1}\hat{\mathbf{r}}^{\top} + \Phi\mathcal{M} D^{-1}\Psi^{\top}$$

with $\mathcal{M} = C\Lambda - \tilde{\Lambda}^{\top} C$, $\qquad \Phi = \varphi\left(C + D^{-1}\{\psi^{\top}\varphi\}\right)^{-1}$,
$\Psi^{\top} = \left(C + D^{-1}\{\psi^{\top}\varphi\}\right)^{-1}\psi^{\top}$, $\quad \hat{\mathbf{q}} = W\{\mathbf{q}\}$, $\quad \hat{\mathbf{r}} = W^{-1,\tau}\{\mathbf{r}\}$.

Solution generating method for (2+1)-dimensional k-cKP hierarchies and their extensions

In a similar way we can dress both operators from the extensions of (2+1)-dimensional k-cKP hierarchies:

$$L_k = \beta_k \partial_{\tau_k} - \sum_{j=0}^{k} u_j D^j - \mathbf{q}\mathcal{M}_0 D^{-1}\mathbf{r}^\top$$
$$M_{n,l} = \alpha_n \partial_{t_n} - \sum_{i=0}^{n} v_i D^i - c_l \sum_{j=0}^{l} \mathbf{q}[j]\mathcal{M}_0 D^{-1}\mathbf{r}^\top[l-j], \quad l = 1, \ldots$$
$$M_{n,l}\{\mathbf{q}\} = c_l (L_k)^{l+1}\{\mathbf{q}\} \qquad M^\tau_{n,l}\{\mathbf{r}\} = c_l (L^\tau_k)^{l+1}\{\mathbf{r}\}$$

Let matrices of functions φ and ψ satisfy:

$$L_k\{\varphi\} = \varphi\Lambda, \qquad L^\tau_k\{\psi\} = \psi\tilde{\Lambda},$$
$$M_{n,l}\{\varphi\} = c_l L_k^{l+1}\{\varphi\}, \qquad M^\tau_{n,l}\{\psi\} = c_l (L^\tau_k)^{l+1}\{\psi\}.$$

Using binary Darboux transformation

$$W = I - \varphi\left(C + D^{-1}\{\psi^\top\varphi\}\right)^{-1} D^{-1}\psi^\top$$

we obtain...

Solution generating method for (2+1)-dimensional k-cKP hierarchies and their extensions

$$\hat{L}_k := WL_kW^{-1} = \beta_k\partial_{\tau_k} - \sum_{j=0}^{k}\hat{u}_jD^j - \hat{\mathbf{q}}\mathcal{M}_0D^{-1}\hat{\mathbf{r}}^\top + \Phi\mathcal{M}_1D^{-1}\Psi^\top,$$

$$\hat{M}_{n,l} := WM_{n,l}W^{-1} = \alpha_n\partial_{t_n} - \sum_{i=0}^{n}\hat{v}_iD^i - c_l\textstyle\sum_{j=0}^{l}\hat{\mathbf{q}}[j]\mathcal{M}_0D^{-1}\hat{\mathbf{r}}^\top[l-j]+$$

$$+c_l\sum_{s=0}^{l}\Phi[s]\mathcal{M}_1D^{-1}\Psi^\top[l-s]$$

with $\mathcal{M}_1 = C\Lambda - \tilde{\Lambda}^\top C, \qquad \Phi = \varphi\left(C + D^{-1}\{\psi^\top\varphi\}\right)^{-1},$
$\Psi^\top = \left(C + D^{-1}\{\psi^\top\varphi\}\right)^{-1}\psi^\top, \quad \hat{\mathbf{q}} = W\{\mathbf{q}\}, \quad \hat{\mathbf{r}} = W^{-1,\tau}\{\mathbf{r}\},$
$\Phi[j] := (\hat{L}_k)^j\{\Phi\} \quad , \Psi[j] := (\hat{L}_k^\tau)^j\{\Psi\}, \quad \hat{\mathbf{q}}[j] = (\hat{L}_k^j)\{\hat{\mathbf{q}}\},$
$\hat{\mathbf{r}}[j] = (\hat{L}_k^j)^\tau\{\hat{\mathbf{r}}\}.$

Conclusions

- New generalizations of the (2+1)-dimensional k-constrained KP hierarchies include matrix DS system, new matrix generalization of the Nizhnik system, (2+1)-dimensional generalizations of the Yajima-Oikawa system, KP equations with self-consistent sources.
- The proposed solution generating method is based on invariant transformations of the integro-differential operators via BDT. It allows to construct solutions of the corresponding nonlinear systems starting with an arbitrary seed solution.

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Thank you for your attention!